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by

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Abstract

This thesis centers mainly on the simple Lie algebras of type C_n over the complex numbers. A completely pointed module is defined to be an infinite dimensional simple highest weight module all of whose weight spaces are one dimensional. We construct examples of completely pointed C_n-modules and give a complete characterization of all such modules. We then study the tensor product of a completely pointed C_n-module with a finite dimensional module and classify all the possible central characters which can appear. Examples indicate that this tensor product is completely reducible and that all the possible central characters actually appear.

Dedicated to:

my wife, Christine

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Chapter 0

Introduction

Let L denote a finite dimensional simple Lie algebra over $\mathbb{C}$, the field of complex numbers, and let $\mathcal{H}$ denote a fixed Cartan subalgebra of L. An important area of current research is to provide explicit constructions of all possible L-modules. A great deal of effort has been expended, yet this problem has only been successfully solved for the simple Lie algebra of type A_1, i.e. the three dimensional Lie algebra of 2×2 complex matrices (see Block [B]). For other simple Lie algebras this problem is still beyond reach.

More recently, attention has shifted to considering a smaller class of L-modules, namely the category $\mathcal{W}(L,\mathcal{H})$ of $(L,\mathcal{H})$ finitely generated weight modules having finite dimensional weight spaces. This problem has been treated by Fernando ([F]), who showed that the classification of all simple modules in the category $\mathcal{W}(L,\mathcal{H})$ reduces to the problem of determining, for simple Lie algebras, all simple finite dimensional modules and all simple torsion free modules with finite dimensional weight spaces.

In view of the classical results of Cartan, Weyl, and Harish-Chandra, the structure of simple finite dimensional modules of a complex simple Lie algebra is well understood. Such modules find numerous applications in physics and chemistry, as well as in other areas of mathematics, and so there is an enormous amount of detailed information in this area.

0. Introduction

Researchers have thus been led to focus on classifying and constructing all simple torsion free L-modules having finite dimensional weight spaces. Fernando ([F]) has observed that the only simple Lie algebras which admit torsion free modules are the classical Lie algebras of types A and C. He also provided an explicit construction of certain families of torsion free modules for the Lie algebras of type A and C, namely torsion free modules having all one dimensional weight spaces. Britten and Lemire ([BL]) proved that the examples constucted in [F] exhaust all such simple torsion free modules. In [BFL], Britten, Futorny and Lemire have recently classified all simple torsion free A_2-modules and have shown that any such module can be realized as subquotients of the tensor product of a pointed torsion free A_2-module with a finite dimensional module. This leads us to consider such tensor products for other simple Lie algebras.

We began by focusing our research on the tensor product of a pointed torsion free C_2-module with a finite dimensional module. Although small examples were found to be completely reducible, in general these modules proved quite difficult to work with. Since highest weight modules have properties similar to finite dimensional modules, we located examples of simple highest weight modules which were infinite dimensional and have the property that all weight spaces are one dimensional. We call such modules completely pointed.

Deep inside a completely pointed module, the local structure is similar to that of a pointed torsion free module, and so hopefully, information obtained about completely pointed modules and the corresponding tensor products with finite dimensional modules can give some insight into the pointed torsion free case.

In this thesis we classify all completely pointed C_n-modules and give explicit constructions of such modules. Considering the tensor product of a completely pointed C_n-module M with a finite dimensional module V, we classify all the possible simple highest weight submodules of $M \otimes V$, as well as all the possible central characters which can occur in $M \otimes V$.

We now briefly outline the contents of this thesis. In Chapter 1, we begin by introducing some structure theory of Lie algebras. We define the basic concepts of homomorphisms and ideals, and give some important examples, including $gl(n, \mathbb{C})$ and $sl(n, \mathbb{C})$, as well as the derivation algebra and adjoint representation. We next consider the concept of solvable Lie algebra and introduce semisimplicity. Introducing the Killing form, we state Cartan's criterion for solvability and develop a criterion for the semisimplicity of a Lie algebra. We also obtain the decomposition of a semisimple Lie algebra into a direct sum of simple ideals.

We next develop the theory of modules over a Lie algebra and give some basic results of representation theory, leading up to Weyl's Theorem on the complete reducibility of finite modules. Then we use Weyl's Theorem to obtain an alternate criterion for semisimplicity. After stating the Classification Theorem for simple Lie algebras, we construct the Lie algebras of type C_n and prove that these algebras are simple. After digressing slightly in order to introduce the Jordan decomposition of elements in a linear Lie algebra, we use this to obtain some basic results on finite dimensional representations of the Lie algebra $sl(2, \mathbb{C})$. Next, we introduce the notion of root space decomposition of a semisimple Lie algebra and develop some of the basic properties of this decomposition. By introducing the concept of root system, we develop some geometrical properties of roots, including the notions of Weyl group and simple bases of roots. We then introduce similar geometrical ideas involving weights. Finally, we introduce the notion of a universal enveloping algebra and state the Poincaré-Birkhoff-Witt Theorem.

0. Introduction

Chapter 2 develops aspects of the representation theory of semisimple Lie algebras. We begin by generalizing the concept of weight and introducing weight space decompositions of modules. After developing some basic properties, we prove several equivalent conditions for a module to admit a weight space decomposition. We then decompose the universal enveloping algebra and prove that the zero weight space is a finitely generated subalgebra. Also, we prove additional results regarding weight space decompositions.

We introduce the concepts of maximal vector and standard cyclic module and prove some basic properties. Using these results, we prove that up to isomorphism there is exactly one irreducible standard cyclic module for each highest weight λ. After developing conditions on λ for the irreducible standard cyclic module $V(\lambda)$ to be finite dimensional, we concentrate on presenting some results from character theory. We first introduce the concept of central character and state the important Harish-Chandra Theorem. Then we concentrate on formal characters and develop some properties. We conclude Chapter 2 by introducing the idea of contravariant forms on highest weight modules and we obtain results regarding the tensor product of a standard cyclic module with a finite dimensional module.

We begin Chapter 3 by focusing our attention on the Lie algebras of type C_n. In particular, we give a simple base for the root system, choose a corresponding basis of C_n, and introduce an important representation. After constructing the examples of completely pointed C_n-modules mentioned above, we prove that (up to isomorphism) these are the only such modules. We then prove several results regarding the structure of highest weight vectors in the tensor products $M \otimes V$ and $N \otimes V$ and use them to obtain the aforementioned conditions on the exponents. We use these conditions to prove that no two maximal vectors in $M \otimes V$ or $N \otimes V$ have highest weights which are linked. Finally, we use this result to prove the desired classifications of standard cyclic submodules and central characters.

Finally, in Chapter 4 we summarize our results. We then give several conjectures and offer an example to support them.

Chapter 1

Structure Theory

In this thesis we will be concerned exclusively with the field $\mathbb{C}$ of complex numbers and with Lie algebras whose underlying vector space is finite dimensional over $\mathbb{C}$. Essentially everything holds for Lie algebras over an arbitrary field F of characteristic 0, but for simplicity we will restrict our attention to $\mathbb{C}$. Also, we will let $\mathbb{Z}^+$ denote the set of nonnegative integers.

1.1: Basic Concepts

We begin by defining the fundamental concepts.

Definition 1.1: By a **Lie algebra** L **over** $\mathbb{C}$ we mean a vector space L over $\mathbb{C}$ endowed with an operation $[\cdot,\cdot]: L \times L \to L$ called the **commutator** or **bracket** product such that the following axioms are satisfied:

(1) The commutator product is bilinear.

(2) $[x,x] = 0 \quad$ for all $x \in L$.

(3) $\big[x,[y,z]\big] = \big[[x,y],z\big] + \big[y,[x,z]\big] \quad$ for all $x,y,z \in L$.

Axiom (3) is known as the **Jacobi identity**. It is easy to see that axioms (1) and (2) imply anticommutativity: (2)′ $[x,y] = -[y,x]$. We call a Lie algebra L **abelian** if for all $x,y \in L$, $[x,y] = 0$.

1. Structure Theory

Let us consider a general example of a Lie algebra which arises naturally. Let V be a finite dimensional vector space over $\mathbb{C}$ and as usual let $\operatorname{End} V$ denote the ring of linear transformations mapping V into itself. We define a commutator multiplication on $\operatorname{End} V$ by setting $[x,y] = xy - yx$ for all $x, y \in \operatorname{End} V$. With this operation, $\operatorname{End} V$ becomes a Lie algebra over $\mathbb{C}$, which we call the **general linear algebra** over V. We denote the general linear algebra by $gl(V)$ so as to distinguish it from the associative algebra structure given to $\operatorname{End} V$ by the usual product of composition. If we fix a basis for V, we may identify $gl(V)$ with the set of all $n \times n$ matrices over $\mathbb{C}$ and denote this by $gl(n, \mathbb{C})$. A basis for $gl(n, \mathbb{C})$ is given by $\{E_{ij} \mid 1 \leq i, j \leq n\}$, where E_{ij} is the $n \times n$ matrix having a 1 in the (i,j)-position and 0's elsewhere.

Many fundamental algebraic notions give rise to corresponding ones for Lie algebras. We say that the Lie algebras L and L' over $\mathbb{C}$ are **isomorphic** if there is a vector space isomorphism $\phi: L \to L'$ such that $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$. A **subalgebra** of L is a vector subspace K of L having the property that $[x,y] \in K$ whenever $x, y \in K$. As an example, any subalgebra of $gl(V)$ is called a **linear Lie algebra**. If L is a Lie algebra then a subspace I of L is called an **ideal** of L (denoted $I \trianglelefteq L$) if for each $x \in L$ and each $y \in I$ we have $[x,y] \in I$. Both (0) and L are ideals of L, called the **trivial ideals**; all other ideals are called **proper**. If I and J are ideals of L, then $I + J = \{x + y \mid x \in I, y \in J\}$ and $[I,J] = \{\sum_{i=1}^{n} a_i [x_i, y_i] \mid n \text{ a pos. integer}, a_i \in \mathbb{C}, x_i \in I, y_i \in J\}$ are also ideals of L. Two important examples of ideals of a Lie algebra L are the **center** of L, $Z(L) = \{x \in L \mid [x,y] = 0 \ \forall y \in L\}$, and the **derived algebra** of L, $[L,L] = \{\sum x_i y_i \mid x_i, y_i \in L\}$. Notice that L is abelian iff $Z(L) = L$ iff $[L,L] = (0)$. Finally, if L has no proper ideals, and if also $[L,L] \neq 0$, we call L **simple**. Obviously, if L

is simple then $L = [L, L]$ and $Z(L) = (0)$.

We now introduce two examples of subalgebras of $gl(n, \mathbb{C})$ which will be needed later. Let $sl(n, \mathbb{C})$ denote the set of all matrices in $gl(n, \mathbb{C})$ having trace 0. Since for any $n \times n$ matrices, $\mathrm{Tr}\,(xy) = \mathrm{Tr}(yx)$ and $\mathrm{Tr}\,(x+y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$, $sl(n, \mathbb{C})$ is clearly a subalgebra of $gl(n, \mathbb{C})$. Also, since $I_n \notin sl(n, \mathbb{C})$, we have that $sl(n, \mathbb{C})$ is actually a proper subalgebra of $gl(n, \mathbb{C})$ (provided of course that $n \geq 2$). We can easily find a basis for $sl(n, \mathbb{C})$. Take all matrices E_{ij} with $i \neq j$, along with the matrices of the form $E_{ii} - E_{i+1,i+1}$ $(1 \leq i \leq n-1)$. This gives $n^2 - 1$ matrices which are clearly linearly independent and traceless. Since $\dim sl(n, \mathbb{C}) < \dim gl(n, \mathbb{C}) = n^2$, these matrices form a basis for $sl(n, \mathbb{C})$. We shall refer to this basis as the **standard basis** of $sl(n, \mathbb{C})$. For the other subalgebra, we let $s(n, \mathbb{C})$ denote the set of all scalar multiples of the identity matrix $I_n \in gl(n, \mathbb{C})$. Since the product and sum of two scalar matrices is again a scalar, and since scalar matrices commute, $s(n, \mathbb{C})$ is in fact a one dimensional abelian subalgebra of $gl(n, \mathbb{C})$ with $I_n = \sum_{i=1}^{n} E_{ii}$ as a basis. In addition, it is clear that $sl(n, \mathbb{C}) \cap s(n, \mathbb{C}) = (0)$ and hence, since $\dim sl(n, \mathbb{C}) + \dim s(n, \mathbb{C}) = \dim gl(n, \mathbb{C})$, we must have $gl(n, \mathbb{C}) = sl(n, \mathbb{C}) + s(n, \mathbb{C})$ and the sum is direct. In particular, if $x, x' \in gl(n, \mathbb{C})$, we may write x and x' uniquely as $x = y + z$, $x' = y' + z'$, with $y, y' \in sl(n, \mathbb{C})$ and $z, z' \in s(n, \mathbb{C})$. Hence

$$[x, x'] = [y, y'] + [y, z'] + [z, y'] + [z, z']. \tag{1.2}$$

Since scalar matrices commute with all other matrices, the last three terms in (1.2) are 0, and so $[x, x'] = [y, y'] \in sl(n, \mathbb{C})$. This means that $[gl(n, \mathbb{C}), gl(n, \mathbb{C})] \subseteq sl(n, \mathbb{C})$. In fact, computations involving the basis vectors can be used to show that $sl(n, \mathbb{C})$ is the derived algebra of $gl(n, \mathbb{C})$.

If a Lie algebra L contains a proper ideal I, we may construct the corresponding **quotient algebra**, denoted L/I as follows. Construct the quotient vector space

L/I and define the commutator product by $[x+I, y+I] = [x,y]+I$. This operation is well-defined, since if $x + I = x' + I$ and $y + I = y' + I$, then $x' = x + w$ and $y' = y + z$ for some $w, z \in I$ and hence $[x', y'] = [x,y] + [x,z] + [w,y] + [w,z]$. Since $[x,z], [w,y], [w,z] \in I$ we have that $[x', y'] + I = [x,y] + I$.

The concept of homomorphism also carries over to Lie algebras. For Lie algebras L,L' over $\mathbb{C}$, a linear transformation $\phi: L \to L'$ is called a **homomorphism** (or more simply a **map** when no confusion will arise) if $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$. We have the usual definitions of the kernel and image of the homomorphism ϕ. For the map $\phi: L \to L'$ the **kernel** of ϕ is defined to be $\operatorname{Ker}\phi = \{x \in L \mid \phi(x) = 0\}$, while the **image** of ϕ, denoted $\operatorname{Im}\phi$, is defined by $\operatorname{Im}\phi = \{y \in L' \mid \exists x \in L \text{ such that } \phi(x) = y\}$. We can see immediately that $\operatorname{Ker}\phi$ is an ideal in L and $\operatorname{Im}\phi$ is a subalgebra of L'. To see that the first of these holds, simply make the observation that if $\phi(x) = 0$ and $y \in L$ then $\phi([x,y]) = [\phi(x), \phi(y)] = 0$. The other claim holds since by definition of linear transformation $\operatorname{Im}\phi$ is a subspace of L' and since $\phi([x,y]) = [\phi(x), \phi(y)]$. As usual a homomorphism $\phi: L \to L'$ is called a **monomorphism** if $\operatorname{Ker}\phi = (0)$, an **epimorphism** if $\operatorname{Im}\phi = L'$, and as above, an **isomorphism** if ϕ is both a monomorphism and an epimorphism. Lastly, if I is an ideal of L then corresponding to I is the **canonical homomorphism**: $x \to x + I$ of L onto L/I.

We also get the standard homomorphism theorems, which we state for reference:

<u>Theorem 1.3 (First Isomorphism Theorem):</u> *If $\phi: L \to L'$ is a Lie algebra homomorphism, then $L/\operatorname{Ker}\phi \cong \operatorname{Im}\phi$. Moreover, if I is an ideal of L which is contained in $\operatorname{Ker}\phi$, then there exists a unique homomorphism $\theta: L/I \to L'$ such that the following diagram commutes:*

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ & \searrow^{\pi} & \uparrow \theta \\ & & L/I \end{array}$$

(where π is the canonical map). ∎

Theorem 1.4 (Second Isomorphism Theorem) *If I and J are ideals of L such that $I \subset J$, then J/I is an ideal of L/I and $(L/I)/(J/I)$ is isomorphic to L/J.* ∎

Theorem 1.5 (Third Isomorphism Theorem) *If I and J are ideals of L, then $I \cap J$ is an ideal of I, J is an ideal of $I + J$, and*

$$\frac{I+J}{J} \cong \frac{I}{I \cap J}. \quad ∎$$

If V is a vector space over $\mathbb{C}$ then a homomorphism $\phi: L \to gl(V)$ is called a **representation** of L. If ϕ is a monomorphism then ϕ is said to be **faithful**. As an example, we will consider the adjoint representation.

First a few preliminaries are necessary. An **algebra** over $\mathbb{C}$ is a vector space $\mathfrak{A}$ over $\mathbb{C}$ furnished with a bilinear product $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. A **derivation** of $\mathfrak{A}$ is a linear map ∂ having the property that $\partial(xy) = \partial(x)y + x\partial(y)$. We denote the set of all derivations of $\mathfrak{A}$ by $\mathrm{Der}\,\mathfrak{A}$. Linear combinations of derivations are again derivations, and hence $\mathrm{Der}\,\mathfrak{A}$ is a vector subspace of $\mathrm{End}\,\mathfrak{A}$. Moreover, if $\partial, \partial' \in \mathrm{Der}\,\mathfrak{A}$, then for $x, y \in \mathfrak{A}$ we have:

$$\begin{aligned}
[\partial, \partial'](xy) =& \partial\partial'(xy) - \partial'\partial(xy) \\
=& \partial(\partial'(x)y + x\partial'(y)) - \partial'(\partial(x)y + x\partial(y)) \\
=& \partial\partial'(x)y + \partial'(x)\partial(y) + \partial(x)\partial'(y) + x\partial\partial'(y) \\
& - \partial'\partial(x)y - \partial(x)\partial'(y) - \partial'(x)\partial(y) - x\partial'\partial(y)
\end{aligned}$$

$$=\partial\partial'(x)y - \partial'\partial(x)y + x\partial\partial'(y) - x\partial'\partial(y)$$

$$=[\partial,\partial'](x)y + x[\partial,\partial'](y)$$

and so $\mathrm{Der}\,\mathfrak{A}$ is a subalgebra of $gl(\mathfrak{A})$.

Any Lie algebra L can be viewed as an algebra over $\mathbb{C}$ and so $\mathrm{Der}\,L$ is defined. We consider the following derivations of L. If $x \in L$ we denote by $\mathrm{ad}\,x$ the map $y \to [x,y]$. The Jacobi identity gives immediately that $\mathrm{ad}\,x \in \mathrm{Der}\,L$. A derivation of L which can be written in this form is called an **inner** derivation; all other derivations are called **outer**. The map $\mathrm{ad}: L \to \mathrm{Der}\,L \subseteq gl(L)$ is a representation of L called the **adjoint representation**. Note here that $\mathrm{ad}\,x = 0$ if and only if $x \in \mathrm{Ker}\,\mathrm{ad}$ and hence $\mathrm{Ker}\,\mathrm{ad} = Z(L)$. In particular, if L is a simple Lie algebra, then the map $\mathrm{ad}: L \to gl(L)$ is a Lie algebra monomorphism. Thus any simple Lie algebra is isomorphic to a linear Lie algebra.

1.2: Structure Theory

At the end of Section 1 we saw that any simple Lie algebra was isomorphic to a linear Lie algebra. In fact, this is true for any finite dimensional Lie algebra:

Theorem 1.6 (Ado): *If L is any finite dimensional Lie algebra then there exists a finite dimensional vector space V such that L is isomorphic to a subalgebra of $gl(V)$.*

Proof: (see Jacobson [Jac] §6.2). ∎

For a Lie algebra L we define a series of ideals of L called the **derived series** of L by setting $L^{(0)} = L$ and defining, for $n > 0$,

$$L^{(n+1)} = \left[L^{(n)}, L^{(n)}\right] \trianglelefteq L.$$

L is called **solvable** if $L^{(n)} = (0)$ for some n. We can quickly prove some basic properties of solvable Lie algebras.

__Proposition 1.7:__ *Let L be a Lie algebra.*

(1) If L is solvable, then any subalgebra of L is solvable.

(2) If L is solvable, then so is any homomorphic image.

(3) If $I \trianglelefteq L$, I solvable, and L/I is solvable, then L is solvable.

(4) If I and J are solvable ideals of L, then so is $I + J$.

__Proof:__ (1) If K is a subalgebra of L, then by definition $K^{(i)} \subseteq L^{(i)}$ and hence K is solvable.

(2) If L is solvable, let $\phi \colon L \to M$ be an epimorphism. We show $\phi(L^{(i)}) = M^{(i)}$ by inducting on i, the case $i = 0$ being obvious. Then for $n > 0$,

$$\begin{aligned} M^{(n)} &= \left[M^{(n-1)}, M^{(n-1)}\right] \\ &= \left[\phi(L^{(n-1)}), \phi(L^{(n-1)})\right] \qquad \text{by induction} \\ &= \phi\left(\left[L^{(n-1)}, L^{(n-1)}\right]\right) = \phi(L^{(n)}). \end{aligned}$$

(3) Suppose I and L/I are solvable. Then there is an $n > 0$ such that $(L/I)^{(n)} = (0)$ which implies that $L^{(n)} \subseteq I$. Applying part (1), we have that there is $m > 0$ with

$$\left(L^{(n)}\right)^{(m)} = L^{(m+n)} \subseteq I^{(m)} = (0).$$

Hence L is solvable.

(4) By Theorem 1.5,

$$\frac{I}{I \cap J} \cong \frac{(I+J)}{J}.$$

Hence $(I + J)/J$ is solvable, being a homomorphic image of I. Part (3) now applies to give $I + J$ solvable. ∎

1. Structure Theory

Now let L be any Lie algebra and let S be a maximal solvable ideal in L. If I is another solvable ideal, then by the maximality of S, $S + I = S$, i.e. $I \subseteq S$. Thus S contains every other solvable ideal and hence S is unique. We call S the **radical** of L and denote S by $\mathrm{Rad}\, L$.

We define a Lie algebra L to be **semisimple** if $\mathrm{Rad}\, L = (0)$. Note that any simple Lie algebra is clearly semisimple. However, the converse does not hold, since for example (0) is semisimple and not simple. We also remark that the condition $\mathrm{Rad}\, L = (0)$ is equivalent to demanding that L have no nonzero abelian ideals. To see this, note first that any abelian ideal is solvable and so if $\mathrm{Rad}\, L = (0)$ then L can have no nonzero abelian ideals. Conversely, if $\mathrm{Rad}\, L \neq (0)$ then the last nonzero ideal in the derived series of L is abelian. Moreover, if $I/\mathrm{Rad}\, L$ is a nonzero solvable ideal in $L/\mathrm{Rad}\, L$, then $I \trianglelefteq L$ and so by part (3) of the above proposition, I is a solvable ideal in L which properly contains $\mathrm{Rad}\, L$. This contradiction implies that $L/\mathrm{Rad}\, L$ is semisimple for any L.

Throughout the remainder of this thesis we will consider only semisimple Lie algebras over $\mathbb{C}$. For such a Lie algebra, L, we obtain the following result, which has the consequence that L can be realized as a Lie algebra of upper triangular matrices over $\mathbb{C}$.

Theorem 1.8: (Lie's Theorem) *If V is a nonzero finite dimensional vector space and L is a solvable subalgebra of $gl(V)$, then there exists a nonzero common eigenvector for L.*

Proof: (see Humphreys [Hu 1]; §4.1). ∎

If L is any Lie algebra, we define the **Killing form** on L to be the map

$\mathcal{K}: L \times L \to \mathbb{C}$ given by, for $x, y \in L$,

$$\mathcal{K}(x,y) = \mathrm{Tr}\,(\mathrm{ad}_L x \,\mathrm{ad}_L y).$$

Since trace is symmetric and bilinear, we immediately have that $\mathcal{K}$ is a symmetric, bilinear form on L. $\mathcal{K}$ is also associative in the sense that, for any $x, y, z \in L$,

$$\mathcal{K}([x,y],z) = \mathcal{K}(x,[y,z]).$$

To see this, let x, y, and z be endomorphisms of a finite dimensional vector space. Then $[x,y]z = xyz - yxz$ and $x[y,z] = xyz - xzy$, and so

$$\begin{aligned}\mathrm{Tr}\,([x,y]z) &= \mathrm{Tr}\,((xy)z) - \mathrm{Tr}\,((yx)z) \\ &= \mathrm{Tr}\,(x(yz)) - \mathrm{Tr}\,(y(xz)) \\ &= \mathrm{Tr}\,(x(yz)) - \mathrm{Tr}\,((xz)y) = \mathrm{Tr}\,(x[y,z]).\end{aligned}$$

For any symmetric, bilinear form B on a vector space V, we define the **kernel** of B to be

$$\mathrm{Ker}\,B = \{x \in V \mid B(x,y) = 0 \text{ for all } y \in V\}$$

The form B is called **nondegenerate** if $\mathrm{Ker}\,B = (0)$.

Proposition 1.9: *If L is a Lie algebra with Killing form $\mathcal{K}$, then $\mathrm{Ker}\,\mathcal{K} \trianglelefteq L$.*

Proof: Indeed, if $x \in \mathrm{Ker}\,\mathcal{K}$ and $y \in L$, then for any $z \in L$,

$$\mathcal{K}([x,y],z) = \mathcal{K}(x,[y,z]) = 0. \blacksquare$$

We can immediately use the Killing form to derive an important criterion for semisimplicity. First, however, we need to note two results which give important criteria for solvability.

<u>Theorem 1.10: Cartan's Criterion:</u> *Let L be a subalgebra of $gl(V)$, for some finite dimensional vector space V. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x \in [L, L]$ and $y \in L$. Then L is solvable.*

<u>Proof:</u> (see Humphreys [Hu 1]; §4.3). ∎

<u>Corollary 1.11:</u> *Let L be a Lie algebra such that $\mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$ for all $x \in [L, L]$ and $y \in L$. Then L is solvable.*

<u>Proof:</u> (see Humphreys [Hu 1]; §4.3). ∎

<u>Theorem 1.12:</u> *Let L be a Lie algebra. L is semisimple if and only if its Killing form is nondegenerate on L.*

<u>Proof:</u> Suppose that $\mathrm{Rad}\, L = (0)$ and let $S = \mathrm{Ker}\, \mathcal{K}$. By definition of S, we have that $\mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$ for all $x \in S$ and $y \in L$. In particular this holds for all $y \in [S, S]$. Hence Corollary 1.11 applies to give that S is solvable. Thus $S \subseteq \mathrm{Rad}\, L = (0)$, so $S = (0)$, as required.

Conversely, if $S = 0$, it suffices to show that L has no nonzero abelian ideals. Suppose I is an abelian ideal of L. Take $x \in I$ and $y \in L$. Then

$$(\mathrm{ad}\, x\, \mathrm{ad}\, y)^2 = \mathrm{ad}\, x\, \mathrm{ad}\, y\, \mathrm{ad}\, x\, \mathrm{ad}\, y = 0.$$

Hence $\mathrm{ad}\, x\, \mathrm{ad}\, y$ is a nilpotent endomorphism on L and so its trace is 0. In other words, $\mathcal{K}(x, y) = 0$, and since y was an arbitrary element of L, we conclude that $x \in S$, i.e. $I \subseteq S = (0)$. ∎

We say that a Lie algebra L is the direct sum of ideals $I_1, I_2, \ldots, I_n$, denoted $L = \sum_{i=1}^{n} \oplus I_i$, if L is the vector space direct sum of the I_i's. If $i \neq j$, then this definition implies that $[I_i, I_j] \subseteq I_i \cap I_j = (0)$.

Theorem 1.13: *If L is a nonzero semisimple Lie algebra, then L can be written (in an essentially unique way) as a direct sum $L = \sum_{i=1}^{n} \oplus L_i$, where the L_i are simple ideals of L.*

Proof: If I is an arbitrary ideal of L, we let

$$I^{\perp} = \{ x \in L \mid \mathcal{K}(x,y) = 0 \text{ for all } y \in I \}.$$

As in the proof of Proposition 1.9, the associativity of $\mathcal{K}$ implies that $I^{\perp} \trianglelefteq L$. Also, $I \cap I^{\perp}$ is an ideal of L and so, treating $I \cap I^{\perp}$ as a Lie algebra, we may apply Cartan's criterion to get that $I \cap I^{\perp}$ is solvable and hence $I \cap I^{\perp} = (0)$. Since $\dim I + \dim I^{\perp} = \dim L$, we must have $L = I \oplus I^{\perp}$.

We proceed by inducting on $\dim L$. If L has no proper ideals then L is already simple and we are done. Otherwise, let L_1 be a minimal ideal of L. Since every ideal of L_1 is also an ideal of L, L_1 is semisimple. By minimality, L_1 is simple. Similarly, $L_1^{\perp}$ is also semisimple and as above $L = L_1 \oplus L_1^{\perp}$. By induction, L_1 decomposes into a direct sum of simple ideals, and hence we have the required decomposition of L.

To see that this decomposition is unique (up to the order of the factors), let I be a simple nonzero ideal in L. Then $[I, L]$ is an ideal of L contained in I. Also, since L is semisimple, $Z(L) = (0)$ and hence $[I, L] \neq (0)$. Simplicity then forces $[I, L] = I$. On the other hand,

$$I = [I, L] = [I, \sum_{i=1}^{n} \oplus L_i] = \sum_{i=1}^{n} \oplus [I, L_i].$$

Again by simplicity, $I = [I, L_i]$ for some i and hence $I \subset L_i$. Hence $I = L_i$, as required. ∎

As a result, we immediately conclude:

Corollary 1.14: *If L is a semisimple Lie algebra, then $L = [L, L]$ and all ideals and homomorphic images of L are semisimple. In addition, by Theorem 1.13, every ideal of L is a direct sum of simple ideals of L.* ∎

Another important consequence of Theorem 1.12, which will be needed later, concerns derivations of a semisimple Lie algebra. First we need a lemma.

Lemma 1.15: *Let I be an ideal of L, $\mathcal{K}$ the Killing form of L, and $\mathcal{K}_I$ the Killing form of I. Then $\mathcal{K}_I = \mathcal{K}\downarrow_{I\times I}$.*

Proof: We first recall that if W is a subspace of the finite dimensional vector space V, and if $\varphi \in \operatorname{End} V$ is such that φ maps V into W, then $\operatorname{Tr}\varphi = \operatorname{Tr}(\varphi|_W)$. So if $x, y \in I$, then $\operatorname{ad} x \operatorname{ad} y$ is an endomorphism of L which maps L into I. Hence

$$\mathcal{K}(x, y) = \operatorname{Tr}((\operatorname{ad} x \operatorname{ad} y)|_I) = \operatorname{Tr}(\operatorname{ad}_I x \operatorname{ad}_I y) = \mathcal{K}_I(x, y). \blacksquare$$

Theorem 1.16: *If L is semisimple, then* $\operatorname{ad} L = \operatorname{Der} L$

Proof: Since L is semisimple, we must have $Z(L) = 0$, and hence the map $L \mapsto \operatorname{ad} L$ is a Lie algebra isomorphism. More specifically, Theorem 1.12 implies that the Killing form on $L' = \operatorname{ad} L$ is nondegenerate. If $x \in L$ and $\partial \in \operatorname{Der} L$, then

$$[\partial, \operatorname{ad} x] = \operatorname{ad}(\partial x). \tag{1.17}$$

But the lemma implies that $\mathcal{K}_{L'}$ is the restriction of the Killing form $\mathcal{K}$ on $\operatorname{Der} L$ to $L' \times L'$. Specifically, if $I = L'$ is the subspace of $\operatorname{Der} L$ which is orthogonal to L' under $\mathcal{K}$, then the nondegeneracy of $\mathcal{K}_{L'}$ implies that $I \cap L' = (0)$. But I and L' are ideals of $\operatorname{Der} L$, and so $[I, L'] = 0$. Hence if $\partial \in I$, then (1.17) implies that for all $x \in L$, $\operatorname{ad}(\partial x) = 0$. Since ad is 1-1, $\partial x = 0$ for $x \in L$. Hence we can conclude that $\partial = 0$ and so $I = (0)$, i.e. $\operatorname{Der} L = L' = \operatorname{ad} L$. ∎

1.3: Modules and Complete Reducibility

We begin with some terminology. By an **L-module**, we mean a vector space V together with a bilinear map $L \times V \mapsto V$ (denoted by $(x, v) \mapsto x \cdot v$) such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \text{for all } x, y \in L \text{ and } v \in V.$$

A subspace W of V which is closed under the action of L is called a **submodule** of V (Notation: $W \leq V$). A submodule W of V is called **proper** if $W \neq (0), V$.

A couple of important constructions of L-modules need to be considered. If V and W are any L-modules, we construct the **direct sum** $V \oplus W$ by constructing the vector space direct sum and defining the action of L on $V \oplus W$ by setting, for $x \in L$ and $(v, w) \in V \oplus W$, $x \cdot (v, w) = (x \cdot v, x \cdot w)$. Now let V and W be L-modules with bases $\{v_i\}$ and $\{w_j\}$, respectively. We define the vector space tensor product (over $\mathbb{C}$) of V and W, denoted $V \otimes W$, to be the linear span of all vectors of the form $v \otimes w$ $(v \in V, w \in W)$, where these vectors satisfy, for all $c, d \in \mathbb{C}$, $v_1, v_2 \in V$, and $w_1, w_2 \in W$,

$$(cv_1 + v_2) \otimes w_1 = c(v_1 \otimes w_1) + v_2 \otimes w_1$$

$$v_1 \otimes (dw_1 + w_2) = d(v_1 \otimes w_1) + v_1 \otimes w_2$$

A basis for $V \otimes W$ is given by the set of all vectors of the form $v_i \otimes w_j$, where v_i is an element in our chosen basis for V and w_j is a basis vector for W. We make $V \otimes W$ into an L-module by defining, for $x \in L$ and $v \otimes w \in V \otimes W$, $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$. To see that $V \otimes W$ is actually an L-module under this action, we need only check the bracket condition. In particular, for

$x, y \in L$ and $v \otimes w \in V \otimes W$:

$$\begin{aligned}
[x,y]\cdot(v\otimes w) &= ([x,y]\cdot v)\otimes w + v\otimes([x,y]\cdot w) \\
&= (x\cdot(y\cdot v) - y\cdot(x\cdot v))\otimes w \\
&\qquad + v\otimes(x\cdot(y\cdot w) - y\cdot(x\cdot w)) \\
&= (x\cdot(y\cdot v)\otimes w + v\otimes x\cdot(y\cdot w)) \\
&\qquad - (y\cdot(x\cdot v)\otimes w + v\otimes y\cdot(x\cdot w)) \\
&= (x\cdot y - y\cdot x)\cdot(v\otimes w).
\end{aligned}$$

Note that the construction of $V \otimes W$ is independent of the of the bases chosen for V and W.

We first note that the notions of representations and modules are in fact equivalent. Indeed, if $\rho\colon L \mapsto \mathrm{gl}(V)$ is a representation of L, then we may view V as an L-module using the action

$$x \cdot v = \rho(x)(v) \tag{1.18}$$

and conversely, given an L-module V, equation (1.18) defines a representation $\rho\colon L \mapsto \mathrm{gl}(V)$.

For L-modules V and W, a linear map $\phi\colon V \to W$ is called an L-**module homomorphism** if for any $v \in V$ and $x \in L$ we have that $x \cdot \phi(v) = \phi(x \cdot v)$. As in section 1.1, we can make use of the usual prefixes, i.e. mono-, epi-, etc. In particular, when the homomorphism ϕ is bijective we call ϕ an **isomorphism** of L-modules. In this case we say that V and W are **equivalent** (or that V and W afford **equivalent representations** of L).

We call an L-module V **irreducible** (or **simple**) if the only submodules W of V are $W = (0)$ and $W = V$. We say that V is **indecomposable** if it is the case that V cannot be written as the direct sum $W \oplus W'$ of two proper submodules W and W'. Finally, V is **completely reducible** if for every submodule W of V there exists a submodule W' of V such that $V = W \oplus W'$.

Our goal in this section is to introduce an important theorem of Weyl about the complete reducibility of finite dimensional modules. We first state the following Lemma, which has uses in many areas of representation theory.

Lemma 1.19 (Schur's Lemma): *If $\rho \colon L \to gl(V)$ is an irreducible representation of L on a finite dimensional vector space V and $T \in gl(V)$ is such that for all $x \in L$, $[T, \rho(x)] = 0$, then T is a scalar multiple of the identity on V.* ∎

We now wish to construct an important element for a faithful finite dimensional representation of L. So let ρ be such a representation. Define a map $\beta \colon L \times L \to \mathbb{C}$ by $\beta(x, y) = \mathrm{Tr}(\rho(x)\rho(y))$ for all $x, y \in L$. As with the Killing form, β is a symmetric, bilinear, associative form on L. Also, since ρ is faithful, β is nondegenerate. Let $\{x_i\}$ and $\{y_i\}$ be dual bases of L with respect to β, i.e., $\beta(x_i, y_j) = \delta_{ij}$. We define the **Casimir element** with respect to ρ, c_ρ, by setting $c_\rho = \sum_{i=1}^n \rho(x_i)\rho(y_i)$. An important property of c_ρ is that it commutes with all $\rho(x)$:

Proposition 1.20: *If ρ and c_ρ are as above, then for each $x \in L$, $[\rho(x), c_\rho] = 0$.*

Proof: For $x \in L$, we have $[x, x_i] = \sum_{j=1}^n a_{ij}\, x_j$ and $[x, y_i] = \sum_{j=1}^n b_{ij}\, y_j$. Now

$$a_{ki} = \beta\big([x, x_k], y_i\big) = -\beta\big(x_k, [x, y_i]\big) = -b_{ik}.$$

Hence, using the fact that $\operatorname{ad}\rho(x)$ is a derivation,

$$\begin{aligned} [\rho(x), c_\rho] &= \sum_i [\rho(x), \rho(x_i)\rho(y_i)] \\ &= \sum_i [\rho(x), \rho(x_i)]\rho(y_i) + \sum_i \rho(x_i)[\rho(x), \rho(y_i)] \end{aligned}$$

$$\begin{aligned}
&= \sum_i \rho([x,x_i])\rho(y_i) + \sum_i \rho(x_i)\rho([x,y_i]) \\
&= \sum_{i,j} a_{ij}\rho(x_j)\rho(y_i) + \sum_i \rho(x_i)\rho([x,y_i]) \\
&= \sum_j \rho(x_j)\rho\Big(-\sum_j b_{ji}(y_i)\Big) + \sum_i \rho(x_i)\rho([x,y_i]) \\
&= \sum_j \rho(x_j)\rho\big(-[x,y_i]\big) + \sum_i \rho(x_i)\rho([x,y_i]) \\
&= -\sum_j \rho(x_j)\rho([x,y_j]) + \sum_i \rho(x_i)\rho([x,y_i]) = 0. \blacksquare
\end{aligned}$$

Note that if V is an irreducible L-module (i.e. $\rho\colon L \to gl(V)$ is an irreducible representation), then applying Schur's Lemma and the previous proposition, we have $c_\rho = \lambda I_V$. Hence $\operatorname{Tr} c_\rho = \lambda \dim V$. On the other hand,

$$\begin{aligned}
\operatorname{Tr} c_\rho &= \sum_{i=1}^{\dim L} \operatorname{Tr}(\rho(x_i)\,\rho(y_i)) \\
&= \sum_{i=1}^{\dim L} \beta(x_i,y_i) = \dim L
\end{aligned}$$

Therefore, $c_\rho = \left(\frac{\dim L}{\dim V}\right) I_V$.

We need one more small lemma, and then we can state Weyl's Theorem.

<u>Lemma 1.21:</u> *Let $\rho\colon L \to gl(V)$ be a representation of a semisimple Lie algebra L. Then $\rho(L) \subset sl(V)$. In particular, L acts trivially on any one dimensional L-module.*

<u>Proof:</u> The second statement follows directly from the first. To prove the first, note that since $L = [L,L]$, and $sl(V) = [gl(V), gl(V)]$, we have

$$\begin{aligned}
\rho(L) = \rho([L,L]) &= \big[\rho(L),\rho(L)\big] \\
&\subset \big[gl(V), gl(V)\big] = sl(V). \blacksquare
\end{aligned}$$

<u>Theorem 1.22 (Weyl):</u> *Let $\rho: L \to gl(V)$ be a finite dimensional representation of a semisimple Lie algebra. Then ρ is completely reducible.*

<u>Proof:</u> (see [Hu 1], §6.3). ∎

We now are in the position to give an alternate criterion for semisimplicity, which is easier to use in practice than the nondegeneracy of the Killing form. First we need a definition. A Lie algebra L is said to be **reductive** if $\operatorname{Rad} L = Z(L)$. Clearly, abelian and semisimple Lie algebras are reductive. An example which turns out to be not quite so trivial is $gl(V)$.

<u>Theorem 1.23:</u> *(a) If L is a reductive Lie algebra, then $L = [L, L] \oplus Z(L)$ and $[L, L]$ is semisimple.*
(b) Let V be finite dimensional and let $L \subseteq gl(V)$ be a nonzero Lie algebra which acts irreducibly on V. Then L is reductive and $\dim Z(L) \leq 1$. Moreover, if $L \subseteq sl(V)$, then L is semisimple.

<u>Proof:</u> (a) Suppose L is reductive. If L is abelian the result is trivial, so assume that L is nonabelian. Then $L' = L/Z(L)$ is semisimple. In addition, $\operatorname{ad} L \cong \operatorname{ad} L'$ acts completely reducibly on L, by Weyl's Theorem. Hence we can write $L = I \oplus Z(L)$ for some ideal I. Also,

$$[L, L] = [I, I] + [I, Z(L)] + [Z(L), I] + [Z(L), Z(L)] = [I, I] \subseteq I$$

But if ν denotes the canonical map $L \to L'$, then $\nu([L, L]) = L' = \nu(I)$. Thus $L = [L, L] \oplus Z(L)$. Finally, $[L, L] \cong L'$ is semisimple.
(b) Set $S = \operatorname{Rad} L$. Using Lie's Theorem (1.8), there exists $v \in V$ such that for all $s \in S$, $s \cdot v = \lambda(s)v$. For any $x \in L$, $[s, x] \in S$ and hence

$$s \cdot (x \cdot v) = \lambda(s)x \cdot v + \lambda\big([s, x]\big)v \tag{1.24}$$

We first show that for $s \in S$ and $x \in L$, $\lambda([x, s]) = 0$. L acts irreducibly on V, and so in particular we may obtain any element of V by repeatedly applying elements of L to v and taking linear combinations. Hence (1.24) implies that for a suitable choice of basis for V, the matrices of all $s \in S$ will be triangular, with the only nonzero diagonal entry being $\lambda(s)$. But any commutator of the form $[s, x] \in S$ has trace 0 and hence λ vanishes on $[S, L]$.

Again using (1.24), we get that $s \in S$ acts diagonally on V as the scalar $\lambda(s)$. In particular, $S = Z(L)$ and hence L is reductive. Moreover, since S acts as scalar matrices, $\dim S \leq 1$.

Finally, if $L \subseteq sl(V)$, then since the only scalar in $sl(V)$ is 0, $S = (0)$ and L is semisimple. ∎

As an example of the use of Theorem 1.23, we use this result to prove that $sl(n, \mathbb{C})$ is semisimple. We saw in section 1.1 that $gl(n, \mathbb{C}) = sl(n, \mathbb{C}) + s(n, \mathbb{C})$, where the sum is direct. Since $gl(n, \mathbb{C})$ acts irreducibly on $V = \mathbb{C}^n$, clearly $sl(n, \mathbb{C})$ does as well. Hence by Theorem 1.23 $sl(n, \mathbb{C})$ is semisimple.

1.4: The Classification Theorem; Lie Algebras of Type C_n

In this section, we classify all finite dimensional simple Lie algebras and construct those Lie algebras which are of interest to us, namely those of type C_n.

Theorem 1.25 (Classification Theorem): *Let L be a simple finite dimensional Lie algebra. Then either L belongs to one of the families*

(i) A_n (for $n \geq 1$);

(ii) B_n (for $n \geq 3$);

(iii) C_n (for $n \geq 2$);

(iv) D_n *(for* $n \geq 4$*);*

or L *is one of the Lie algebras of type* E_6, E_7, E_8, F_4, *or* G_2.

Proof: For the proof of this result, as well as the construction of these Lie algebras, see [Jac], Ch. IV, §4–6. ∎

We now construct the Lie algebras of type C_n, for $n \geq 2$. Consider the vector space $\mathbb{C}^{2n}$ and let $B = \{e_1, \ldots, e_{2n}\}$ be the standard basis. Relative to this basis, we define a nondegenerate skew-symmetric form f on $\mathbb{C}^{2n}$ by the matrix $s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, i.e. for $v, w \in \mathbb{C}^{2n}$ we define $f(v,w) = v^t s w$. Let $sp(2n,\mathbb{C})$ denote the set of all matrices $x \in gl(2n,\mathbb{C})$ satisfying $sx = -x^t s$. To see that $sp(2n,\mathbb{C})$ is a subalgebra of $gl(2n,\mathbb{C})$, let $x, y \in sp(2n,\mathbb{C})$. Then

$$\begin{aligned} s\,[x,y] &= s\,x\,y - s\,y\,x \\ &= -x^t\,s\,y + y^t\,s\,x \\ &= x^t\,y^t\,s - y^t\,x^t\,s \\ &= -(x\,y - y\,x)^t\,s \\ &= -[x,y]^t\,s \end{aligned}$$

and so $[x,y] \in sp(2n,\mathbb{C})$. If we let $x = \begin{pmatrix} m & p \\ q & r \end{pmatrix}$ for $m,p,q,r \in gl(n,\mathbb{C})$, then the condition $sx = -x^t s$ is equivalent to requiring that $p^t = p$, $q^t = q$, and $m^t = -r$. Note that this final condition implies that $\mathrm{Tr}(x) = 0$ and hence $sp(2n,\mathbb{C})$ is a subalgebra of $sl(2n,\mathbb{C})$. We call $C_n = sp(2n,\mathbb{C})$ the **symplectic algebra** of rank n. We now find a basis of C_n. First take the diagonal matrices $E_{ii} - E_{n+i,n+i}$ $(1 \leq i \leq n)$ and the matrices $E_{ij} - E_{n+j,n+i}$, $(1 \leq i \neq j \leq n)$. This takes care of m and r. For p, we take the matrices $E_{i,n+i}$ $(1 \leq i \leq n)$ and $E_{i,n+j} + E_{j,n+i}$,

$(1 \leq i < j \leq n)$, and similarly for q. This gives a total of $2n^2 + n$ matrices and these clearly form a basis for C_n.

Finally, we prove that the Lie algebras of type C_n are simple.

Theorem 1.26: *The Lie algebras of type C_n $(n \geq 2)$ are simple.*

Proof: Let I be a nonzero ideal in C_n and let $x \in I$ such that $x \neq 0$. Writing x in terms of the basis for C_n constructed above, we have that

$$\begin{aligned} x = \sum_{i=1}^{n}\big(a_i\,(E_{ii} - E_{n+i,n+i}) + b_i\,E_{i,n+i} + c_i\,E_{n+i,i}\big) \\ + \sum_{j \neq k} d_{jk}\,(E_{jk} - E_{n+k,n+j}) \\ + \sum_{j<k}\big(e_{jk}\,(E_{j,n+k} + E_{k,n+j}) + f_{jk}\,(E_{n+j,k} + E_{n+k,j})\big) \end{aligned}$$

where at least one coefficient is nonzero.

We prove first that for some i, $E_{ii} - E_{n+i,n+i} \in I$. Suppose first that $a_i \neq 0$ for some i. Then

$$\begin{aligned} x' = [x, E_{i,n+i}] = 2a_i\,E_{i,n+i} - c_i(E_{ii} - E_{n+i,n+i}) \\ - \sum_{j \neq i} d_{ji}\,(E_{i,n+j} + E_{j,n+i}) \\ - \sum_{i<k} f_{ik}\,(E_{ik} - E_{n+k,n+i}) \in I, \end{aligned}$$

and so

$$\begin{aligned} x'' = [x', E_{n+i,i}] = 2a_i\,(E_{ii} - E_{n+i,n+i}) + 2c_i\,E_{n+i,i} \\ + \sum_{j \neq i} d_{ji}\,(E_{ji} - E_{n+i,n+j}) \\ + \sum_{i<k} f_{ik}\,(E_{n+i,k} + E_{n+k,i}) \in I. \end{aligned}$$

This implies that

$$x''' = [x'', E_{n+i,i}] = -4a_i E_{n+i,i} \in I.$$

Hence $E_{n+i,i} \in I$ and so $[E_{i,n+i}, E_{n+i,i}] = E_{ii} - E_{n+i,n+i} \in I$.

If $b_i \neq 0$ for some i, then

$$x' = [E_{i,n+i}, x] = b_i (E_{ii} - E_{n+i,n+i}) + \cdots.$$

Since $b_i \neq 0$, the first case applies to x' to give $E_{ii} - E_{n+i,n+i} \in I$.

Similarly, if $c_i \neq 0$ for some i, then

$$x' = [x, E_{n+i,i}] = c_i (E_{ii} - E_{n+i,n+i}) + \cdots,$$

and again we get $E_{ii} - E_{n+i,n+i} \in I$.

If $d_{jk} \neq 0$ for some indices $j \neq k$, then

$$\begin{aligned} x' = [x, E_{k,j} - E_{n+j,n+k}] &= d_{jk} (E_{jj} - E_{n+j,n+j}) \\ &\quad - d_{jk} (E_{kk} - E_{n+k,n+k}) + \cdots. \end{aligned}$$

In particular, the argument used in the first case can be used (with $i = j$ and $i = k$, respectively) to give that $E_{jj} - E_{n+j,n+j}, E_{kk} - E_{n+k,n+k} \in I$.

In a similar manner, if e_{jk} or f_{jk} is nonzero, we can compute $[x, E_{n+j,k} + E_{n+k,j}]$ and $[E_{j,n+k} + E_{k,n+j}, x]$, respectively. Again using the first case, we get $E_{jj} - E_{n+j,n+j}, E_{kk} - E_{n+k,n+k} \in I$. Hence if $x \in I$, with $x \neq 0$, we necessarily have $E_{ii} - E_{n+i,n+i} \in I$ for some i.

Now, if $j \neq i$,

$$[E_{ii} - E_{n+i,n+i}, E_{i,n+j} + E_{j,n+i}] = E_{i,n+j} + E_{j,n+i} \in I,$$

and so

$$[E_{i,n+j} + E_{j,n+i}, E_{n+i,j} + E_{n+j,i}] = E_{ii} - E_{n+i,n+i} + E_{jj} - E_{n+j,n+j} \in I.$$

Hence we have that $E_{jj} - E_{n+j,n+j} \in I$. Thus $E_{ii} - E_{n+i,n+i} \in I$ for $1 \leq i \leq n$.

Finally, consider the equations

$$\begin{aligned}
E_{i,n+i} &= [E_{ii} - E_{n+i,n+i}, \frac{1}{2}E_{i,n+i}] \\
E_{n+i,i} &= [E_{ii} - E_{n+i,n+i}, -\frac{1}{2}E_{n+i,i}] \\
E_{ij} - E_{n+j,n+i} &= [E_{ii} - E_{n+i,n+i}, E_{ij} - E_{n+j,n+i}] \\
E_{i,n+j} + E_{j,n+i} &= [E_{ii} - E_{n+i,n+i}, E_{i,n+j} + E_{j,n+i}] \\
E_{n+i,j} + E_{n+j,i} &= [E_{ii} - E_{n+i,n+i}, -E_{n+i,j} + E_{n+j,i}]
\end{aligned}$$

By considering appropriate choices of i and j, these equations give that all basis elements are in I and hence $I = C_n$. ∎

1.5: The Jordan Decomposition of an Endomorphism

We digress slightly in order to introduce an important decomposition of elements in a Lie algebra, which will be used in the next section to decompose finite dimensional $sl(2,\mathbb{C})$-modules.

We first consider a decomposition of endomorphisms of a finite dimensional vector space V. To this end, let $x \in \operatorname{End} V$. We call x **semisimple** if x is diagonalizable. Also, we recall that x is called **nilpotent** if there exists $n \geq 0$ such that $x^n = 0$.

__Proposition 1.27:__ *Let V be a finite dimensional vector space over $\mathbb{C}$ and let $x \in \operatorname{End} V$.*

(1) There exist unique $x_s, x_n \in \operatorname{End} V$ such that $x = x_s + x_n$ with x_s semisimple, x_n nilpotent, and such that $[x_s, x_n] = 0$.
(2) x_s and x_n commute with any endomorphism which commutes with x.
(3) If $W_1 \subseteq W_2 \subseteq V$ are subspaces such that $x(W_2) \subseteq W_1$, then $x_s(W_2) \subseteq W_1$ and $x_n(W_2) \subseteq W_1$.

Proof: (see [Hu 1], §4.2). ∎

The decomposition $x = x_s + x_n$ in Proposition 1.27 is called the **Jordan decomposition** of x. We also state an additional useful fact.

Lemma 1.28: *Let $\mathfrak{A}$ be a finite dimensional $\mathbb{C}$-algebra. Then $\operatorname{Der}\mathfrak{A}$ contains the semisimple and nilpotent parts (in $\operatorname{End}\mathfrak{A}$) of all its elements.*

Proof: (see [Hu 1], §4.2). ∎

Using Theorem 1.16, we can introduce an abstract Jordan decomposition of elements in an arbitrary semisimple Lie algebra L. By Theorem 1.16, $\operatorname{ad} L = \operatorname{Der} L$, and so by Lemma 1.28, $\operatorname{ad} L$ contains the semisimple and nilpotent parts of all its elements. Also, the map $L \to \operatorname{ad} L$ is 1-1 and so each $x \in L$ determines unique elements $s, n \in L$ such that $\operatorname{ad} x = \operatorname{ad} s + \operatorname{ad} n$ is the Jordan decomposition of $\operatorname{ad} x$ in $\operatorname{End} L$ (i.e. $\operatorname{ad} s$ is semisimple and $\operatorname{ad} n$ is nilpotent). Since $\operatorname{ad} s + \operatorname{ad} n = \operatorname{ad}(s+n)$, we must have $x = s + n$. Also, since $\operatorname{ad}[s, n] = [\operatorname{ad} s, \operatorname{ad} n] = 0$ by Proposition 1.27, we have $[s, n] = 0$.

It turns out that if L is in addition a linear Lie algebra, then the abstract Jordan decomposition thus constructed coincides with the usual Jordan decomposition of endomorphisms.

Theorem 1.29: *Let $L \subset gl(V)$ be a semisimple linear Lie algebra, with V finite* *dimensional. Then L contains the semisimple and nilpotent parts in $gl(V)$ of all its elements. In particular, the abstract and usual Jordan decompositions in L coincide.*

Proof: Since each type of Jordan decomposition is unique, the second statement follows directly from the first.

Take $x \in L$ and let $x = x_s + x_n$ be the usual Jordan decomposition of x in $gl(V)$. Set $N = \{ z \in gl(V) \mid [z, L] \subseteq L \}$. If $\mathrm{ad} = \mathrm{ad}_{gl(V)}$, then since $\mathrm{ad}\, x(L) \subseteq L$, Proposition 1.27 (3) implies that $\mathrm{ad}\, x_s(L) \subseteq L$ and $\mathrm{ad}\, x_n(L) \subseteq L$. Hence $x_s, x_n \in N$. For any submodule W of V, we define

$$L_W = \{ y \in gl(V) \mid y(W) \subseteq W \text{ and } \mathrm{Tr}(y\downarrow_W) = 0 \}$$

Since $L = [L, L] \subseteq sl(V)$, L consists of traceless endomorphisms, and hence $L \subseteq L_W$ for all such W. Let $L' = N \cap (\cap_{W \leq V} L_W)$. L' is clearly a subalgebra of N which includes L as an ideal. Notice also that no L_W contains any scalars, and so in particular L' does not contain the scalars. In addition, if $x \in L$, then $x_s, x_n \in L_W$. Hence $x_s, x_n \in L'$.

It remains only to prove that $L = L'$. We can view L' as a finite dimensional L-module, and so Weyl's Theorem implies that we can write $L' = L + M$ for some L-module M, where the sum is direct. Since $L' \subseteq N$, we have $[L, L'] \subseteq L$, and so L acts trivially on M. Let W be an irreducible submodule of V. If $y \in M$ then $[L, y] = 0$, and hence by Schur's Lemma, y acts on W as a scalar. But $y \in L_W$ and so $\mathrm{Tr}(y\downarrow_W) = 0$. Hence $y\downarrow_W = 0$. Since V is completely reducible (Weyl's Theorem), V can be written as a direct sum of irreducible L-modules and so $y = 0$. Therefore $M = (0)$ and $L = L'$. ∎

Corollary 1.30: *Let L be a semisimple Lie algebra and let $\phi \colon L \to gl(V)$ be a finite* *dimensional representation of L. If $x = s + n$ is the abstract Jordan decomposition of $x \in L$, then $\phi(x) = \phi(s) + \phi(n)$ is the usual Jordan decomposition of $\phi(x)$.*

Proof: Since L is spanned by the eigenvectors of $\operatorname{ad} s$, the algebra $\phi(L)$ must be spanned by the eigenvectors of $\operatorname{ad}_{\phi(L)}\phi(s)$. Hence $\operatorname{ad}_{\phi(L)}\phi(s)$ is semisimple. Similarly, $\operatorname{ad}_{\phi(L)}\phi(n)$ is nilpotent. Thus $\phi(x) = \phi(s) + \phi(n)$ is the abstract Jordan decomposition of $\phi(x)$ (since $\phi(L)$ is semisimple). The result now follows immediately from Theorem 1.29. ∎

1.6: Finite Dimensional Representations of $sl(2,\mathbb{C})$

In this section we classify all the finite dimensional representations of $sl(2,\mathbb{C})$. We will see later that any semisimple Lie algebra is composed of isomorphic copies of $sl(2,\mathbb{C})$, and so the representation theory of $sl(2,\mathbb{C})$ plays a vital role in the representation theory of semisimple Lie algebras. Recall from section 1.3 that $L = sl(2,\mathbb{C})$ is a semisimple subalgebra of $gl(2,\mathbb{C})$. We saw in Section 1.1 that a basis for L consists of $x = E_{12}$, $y = E_{21}$, and $h = E_{11} - E_{22}$. Multiplication in L is given explicitly by

$$[x,y] = h\,, \qquad [h,x] = 2x\,, \qquad [h,y] = -2y.$$

Let V be a finite dimensional L-module. By Corollary 1.30, since h is semisimple, h acts diagonally on V. Hence we may decompose V as a direct sum of spaces $V_\lambda = \{\, v \in V \mid h \cdot v = \lambda v \,\}$. Whenever $V_\lambda \neq 0$ we call λ a **weight** of h in V and V_λ the corresponding **weight space**. An element $v \in V_\lambda$ for which $x \cdot v = 0$ is called a **maximal vector** of weight λ.

We need a couple of technical lemmas regarding the action of L on V.

Lemma 1.31: *If $v \in V_\lambda$ then $x \cdot v \in V_{\lambda+2}$ and $y \cdot v \in V_{\lambda-2}$.*

Proof: We prove the result for x; a similar argument applies for y. It suffices to note that

$$\begin{aligned} h \cdot (x \cdot v) &= x \cdot (h \cdot v) + [h, x] \cdot v \\ &= \lambda(x \cdot v) + 2(x \cdot v) = (\lambda + 2)(x \cdot v). \ \blacksquare \end{aligned}$$

Note that since V is finite dimensional and the sum $V = \sum_{\lambda \in \mathbb{C}} \oplus V_\lambda$ is direct, there must be $\lambda \in \mathbb{C}$ for which $V_\lambda \neq (0)$, yet $V_{\lambda+2} = (0)$. According to the above Lemma, any nonzero vector in V_λ is maximal.

We now assume that V is an irreducible finite dimensional L-module. Let $v_0 \in V_\lambda$ be a maximal vector of weight λ.

Lemma 1.32: *Define $v_{-1} = 0$ and for each $i > 0$, set $v_i = (1/i!)y^i \cdot v_0$. Then for $i \geq 0$ we have*

(1) $h \cdot v_i = (\lambda - 2i)v_i$

(2) $y \cdot v_i = (i + 1)v_{i+1}$

(3) $x \cdot v_i = (\lambda - i + 1)v_{i-1}$

Proof: Clearly (1) follows from Lemma 1.31 and (2) is just the definition of v_i. To show that (3) holds, we induct on i, the case $i = 0$ being trivial. Note that $x \cdot y \cdot v_{i-1} = i(x \cdot v_i)$ by (2). Also,

$$\begin{aligned} x \cdot y \cdot v_{i-1} &= y \cdot x \cdot v_{i-1} + [x, y] \cdot v_{i-1} \\ &= y \cdot x \cdot v_{i-1} + h \cdot v_{i-1} \\ &= (\lambda - i + 2)y \cdot v_{i-2} + (\lambda - 2(i - 1))v_{i-1} \qquad \text{(induction)} \\ &= (i - 1)(\lambda - i + 2)v_{i-1} + (\lambda - 2(i - 1))v_{i-1} \\ &= i(\lambda - i + 1)v_{i-1} \end{aligned}$$

and so $x \cdot v_i = (\lambda - i + 1)v_{i-1}$. ∎

<u>Theorem 1.33:</u> *Let $L = sl(2,\mathbb{C})$ and let V be an irreducible L-module.*

(1) With respect to h, V is the direct sum of the weight spaces V_μ, with $\mu = m, m-2, m-4, \ldots, -m$, where $\dim V = m+1$ and for each μ, $\dim V_\mu = 1$.

(2) Up to nonzero scalar multiples, V has a unique maximal vector, of highest weight m.

(3) The v_i's chosen in Lemma 1.32 form a basis for V and the equations (1)–(3) explicitly describe the action of L on V. Up to isomorphism there is precisely one irreducible L-module of each possible dimension $m+1$ $(m \geq 0)$.

<u>Proof:</u> Lemma 1.32 part (1) says that the v_i are linearly independent. Since $\dim V < \infty$, there must be m such that $v_m \neq 0$ but $v_{m+1} = 0$. Clearly $v_{m+i} = 0$ for $i > 0$. Parts (1)–(3) also imply that the subspace of V with basis $\{v_0, \ldots, v_m\}$ is a nonzero submodule of V. Hence this must be all of V, since V is irreducible. In addition, the matrices for the actions of x, y, and h can be computed explicitly. The matrix for h is diagonal, while that for x is strictly upper triangular, and that for y is strictly lower triangular.

Also, if $i = m+1$, Lemma 1.32 part (3) implies that $(\lambda - m)v_m = 0$, i.e. $\lambda = m$. Hence the highest weight of V is a nonnegative integer. Moreover, by part (1) of Lemma 1.32, $\dim V_\mu = 1$ if $V_\mu \neq (0)$.

Since V determines λ uniquely, V has exactly one maximal vector, with the exception of nonzero scalar multiples. This takes care of statements (1) and (2). Finally, equations (1)–(3) of Lemma 1.32 can be used to construct an irreducible module of any finite dimension $m+1$ $(m \geq 0)$, so statement (3) holds as well. ∎

1.7: Root Space Decomposition

Throughout this section L will denote a nonzero semisimple Lie algebra. By a **toral subalgebra** T of L we mean a subalgebra of L consisting entirely of ad-semisimple elements (i.e. for every element $x \in T$, $\operatorname{ad} x$ is a semisimple endomorphism).

Lemma 1.34: *Any toral subalgebra T of L is abelian.*

Proof: Fix $x \in T$. Suppose that there exists $y \in T$ such that $[x, y] = ay$ with $a \neq 0$. Then $\operatorname{ad}_T y(x) = -ay$ is an eigenvector of $\operatorname{ad}_T y$ with eigenvalue 0. Since $\operatorname{ad}_T y$ is semisimple, we can find a basis $\{y_i \mid i = 1, \ldots, m\}$ of T consisting of eigenvectors of $\operatorname{ad}_T y$, with $\operatorname{ad}_T y(y_i) = a_i y_i$. Write $x = \sum_{i=1}^{m} c_i y_i$. Then $0 = (\operatorname{ad}_T y)^2 x = \sum_{i=1}^{m} c_i a_i^2 y_i$ and so $c_i a_i^2 = 0$ for all i. Hence $c_i a_i = 0$ for all i and so

$$[y, x] = \operatorname{ad}_T y(x) = \sum_{i=1}^{m} c_i a_i y_i = 0,$$

contrary to our choice of nonzero a. Thus $\operatorname{ad}_T x = 0$, as required. ∎

We define a **Cartan subalgebra** of L to be a maximal toral subalgebra. For the remainder of this section, $\mathcal{H}$ will denote a fixed Cartan subalgebra of L.

According to Lemma 1.34, $\mathcal{H}$ is abelian and so $\operatorname{ad}_L \mathcal{H}$ is a family of commuting semisimple endomorphisms on L. Hence $\operatorname{ad}_L \mathcal{H}$ is simultaneously diagonalizable. Thus we can decompose L into a direct sum $L = \sum_{\alpha \in \mathcal{H}^*} L_\alpha$ of spaces of the form

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)\, x \text{ for all } h \in \mathcal{H}\}.$$

We let Φ denote the set of all nonzero $\alpha \in \mathcal{H}^*$ for which $L_\alpha \neq 0$, and call the elements of Φ the **roots** of L relative to $\mathcal{H}$. It is important to note that since L is finite dimensional, Φ is a finite set. We call the decomposition

$$L = L_0 \oplus \sum_{\alpha \in \Phi} L_\alpha$$

the **root space decomposition** of L with respect to $\mathcal{H}$. Lemma 1.34 implies that $\mathcal{H} \subseteq L_0 = C_L(\mathcal{H})$, the centralizer of $\mathcal{H}$ in L (by definition, $C_L(\mathcal{H}) = \{x \in L \mid [x,h] = 0$ for all $h \in \mathcal{H}\}$). It can be shown (see [Hu 1], §8.1) that in fact $\mathcal{H} = C_L(\mathcal{H})$.

In the remainder of this section we prove some basic properties of the set Φ of roots of L (with respect to $\mathcal{H}$).

Proposition 1.35: *Let L and $\mathcal{H}$ be fixed as above and let Φ be the set of roots of L(relative to $\mathcal{H}$).*

(1) If $\alpha, \beta \in \mathcal{H}^*$, then $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

(2) If $\alpha \in \Phi$ and $x \in L_\alpha$, then $\operatorname{ad} x$ is nilpotent.

(3) If $\alpha, \beta \in \Phi$ with $\alpha + \beta \neq 0$, then $\mathcal{K}(L_\alpha, L_\beta) = 0$.

(4) $\mathcal{K}\downarrow_{L_0}$ is nondegenerate.

Proof: (1) Let $x \in L_\alpha$, $y \in L_\beta$, and $h \in \mathcal{H}$. Then

$$\begin{aligned} \operatorname{ad} h([x,y]) &= \big[h,[x,y]\big] \\ &= \big[[h,x],y\big] + \big[x,[h,y]\big] \\ &= (\alpha(h) + \beta(h))[x,y]. \end{aligned}$$

(2) By finiteness of Φ, this follows directly from (1).

(3) Let $x \in L_\alpha$ and $y \in L_\beta$. Then for $h \in \mathcal{H}$,

$$\begin{aligned} \alpha(h)\,\mathcal{K}(x,y) &= \mathcal{K}([h,x],y) \\ &= -\mathcal{K}(x,[h,y]) = -\beta(h)\,\mathcal{K}(x,y) \end{aligned}$$

and the result follows.

(4) This follows directly from (3) and the fact that $\mathcal{K}$ is nondegenerate on L. ∎

In particular, the computation in the proof of part (3) above implies that $\mathcal{K}(L_0, L_\alpha) = 0$ for all $\alpha \in \Phi$ and hence $\mathcal{K}$ is nondegenerate on $\mathcal{H} = L_0$. Hence for $\alpha \in \Phi$, the Riesz Representation Theorem in analysis implies the existence of $t_\alpha \in \mathcal{H}$ such that for all $h \in \mathcal{H}$, $\mathcal{K}(h, t_\alpha) = \alpha(h)$.

Proposition 1.36: *Φ spans $\mathcal{H}^*$. Moreover, if $\alpha \in \Phi$, then*

(1) $-\alpha \in \Phi$;

(2) for $x \in L_\alpha$ and $y \in L_\beta$, we have $[x,y] = \mathcal{K}(x,y)t_\alpha$;

(3) $\dim[L_\alpha, L_{-\alpha}] = 1$;

(4) $\mathcal{K}(t_\alpha, t_\alpha) = \alpha(t_\alpha) \neq 0$;

(5) there exists a subalgebra of $L_\alpha \oplus \mathcal{H} \oplus L_{-\alpha}$ isomorphic to $sl(2,\mathbb{C})$; and

(6) if $h_\alpha = \frac{2t_\alpha}{\mathcal{K}(t_\alpha, t_\alpha)}$, then $h_{-\alpha} = -h_\alpha$.

Proof: Suppose that there exists $h \in \mathcal{H}$ with the property that $\alpha(h) = 0$ for all $\alpha \in \Phi$. Then $[h, L_\alpha] = \alpha(L_\alpha) = 0$ and since $[h, \mathcal{H}] = 0$, we have $[h, L] = 0$. Thus $h \in Z(L) = (0)$. For the other statements:

(1) This follows directly, since if $\alpha \in \Phi$ and $-\alpha \notin \Phi$ then by part (3) of Proposition 1.35, $\mathcal{K}(L_\alpha, L_\beta) = 0$ for all $\beta \in \Phi$, and as noted in the preamble, $\mathcal{K}(L_\alpha, L_0) = 0$, contrary to the nondegeneracy of $\mathcal{K}$.

(2) Take $x \in L_\alpha$ and $y \in L_{-\alpha}$. Then $[x,y] \in \mathcal{H}$. Hence for any $h \in \mathcal{H}$,

$$\begin{aligned}\mathcal{K}(h,[x,y]) &= \mathcal{K}([h,x],y) = \mathcal{K}(\alpha(h)x,y)\\ &= \alpha(h)\,\mathcal{K}(x,y) = \mathcal{K}(h,t_\alpha)\,\mathcal{K}(x,y) = \mathcal{K}(h,\mathcal{K}(x,y)t_\alpha)\end{aligned}$$

and the result follows from the nondegeneracy of $\mathcal{K}$ on $\mathcal{H}$.

(3) Let $x \in L_\alpha, x \neq 0$. If $\mathcal{K}(x, L_{-\alpha}) = 0$, then $\mathcal{K}(x, L) = 0$. Since $\mathcal{K}$ is nondegenerate on L, there must exist $y \in L_{-\alpha}$ with $\mathcal{K}(x,y) \neq 0$ and hence $[x,y] \neq 0$. Clearly (2) now implies that $[L_\alpha, L_{-\alpha}]$ is one dimensional, with basis t_α.

(4) If $\alpha \in \Phi$ with $\alpha(t_\alpha) = 0$, then choose $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $[x,y] = t_\alpha$. Let S be the Lie algebra spanned by $\{x, y, t_\alpha\}$. We have $[t_\alpha, x] = \alpha(t_\alpha)x = 0$ and, similarly, $[t_\alpha, y] = 0$. Hence $\mathrm{ad}_S\, t_\alpha$ is nilpotent. But $t_\alpha \in \mathcal{H}$ and so $\mathrm{ad}_S\, t_\alpha$ is also semisimple. This means that $\mathrm{ad}_S\, t_\alpha \equiv 0$. In particular, $t_\alpha \in Z(L) = (0)$, contrary to our choice of t_α.

(5) Fix $\alpha \in \Phi$ and let $h_\alpha = \frac{2t_\alpha}{\mathcal{K}(t_\alpha, t_\alpha)}$. Given $x \in L_\alpha$, $x \neq 0$, find $y \in L_{-\alpha}$ such that $\mathcal{K}(x,y) = \frac{2}{\mathcal{K}(t_\alpha, t_\alpha)}$. Then $[x,y] = h_\alpha$. Also,

$$[h_\alpha, x] = \frac{2}{\alpha(t_\alpha)}[t_\alpha, x] = \frac{2\alpha(t_\alpha)}{\alpha(t_\alpha)}x = 2x.$$

Similarly, $[h_\alpha, y] = -2y$. Hence $\mathrm{sp}\{x, y, h_\alpha\}$ is isomorphic to $sl(2,\mathbb{C})$.

(6) This follows directly from the observation that $t_{-\alpha} = -t_\alpha$. ∎

Proposition 1.37: *(1)* $\alpha \in \Phi$ *implies that* $\dim L_\alpha = 1$.

(2) $\mathbb{C}\alpha \cap \Phi = \{\pm\alpha\}$.

(3) $\alpha, \beta \in \Phi$ *implies that* $\beta(h_\alpha) \in \mathbb{Z}$ *and* $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(4) $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.

(5) L is generated as a Lie algebra by $\{L_\alpha \mid \alpha \in \Phi\}$.

Proof: (see [Hu 1], §8.4). ∎

1.8: Root Systems, The Weyl Group, and Simple Bases

Let $(\cdot,\cdot)$ be the usual inner product on $\mathbb{R}^l$. We call a subset Φ of $\mathbb{R}^l$ a **root system** in $\mathbb{R}^l$ if

(1) Φ is finite and spans $\mathbb{R}^l$;

(2) If $\alpha \in \Phi$, $\{\mathbb{R}\alpha\} \cap \Phi = \{\pm\alpha\}$;

(3) If $\alpha, \beta \in \Phi$, then $\beta - \langle\beta,\alpha\rangle\alpha \in \Phi$;

(4) If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) = \langle\beta,\alpha\rangle \in \mathbb{Z}$;

where $\langle\beta,\alpha\rangle = \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

We need to recall a little geometry. A **hyperplane** in $\mathbb{R}^l$ is a subspace of codimension 1. By a **reflection** in $\mathbb{R}^l$ we mean an invertible linear transformation which leaves some hyperplane fixed pointwise and sends any vector orthogonal to that hyperplane to its negative. Any vector $\alpha \in \mathbb{R}^l$ determines a reflection σ_α with reflecting hyperplane $\{\beta \in \mathbb{R}^l \mid (\beta,\alpha) = 0\}$, given explicitly by:

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \langle\beta,\alpha\rangle\alpha.$$

We apply these notions, in particular, to elements of a root system Φ.

For a given root system Φ we define $\mathcal{W}$ to be the group generated by the reflections σ_α ($\alpha \in \Phi$). Axiom (3) implies that $\mathcal{W}$ permutes the set Φ. By (1), we can identify $\mathcal{W}$ with a subgroup of the symmetric group on Φ, and hence $\mathcal{W}$ is finite. We call $\mathcal{W}$ the **Weyl group** of Φ.

If Φ is a root system, we call Δ a **simple base** of Φ if (1) Δ is a basis of $\mathbb{R}^l$, and (2) each root β can be written as $\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$ where the $k_\alpha \in \mathbb{Z}$ are either all nonnegative or all nonpositive.

We call the roots in Δ **simple**. Clearly, $|\Delta| = l$ and the expression in (2) is unique. If all $k_\alpha \geq 0$ we call β a **positive** root (similarly for **negative** roots) and write $\beta \succ 0$ ($\beta \prec 0$). We denote the sets of positive and negative roots with respect to Δ by, respectively, $\Phi^+ = \Phi^+(\Delta)$ and $\Phi^- = \Phi^-(\Delta)$.

For each positive root $\alpha \succ 0$, since $\dim L_\alpha = 1$ by Proposition 1.37 (a), we may choose a basis vector $X_\alpha \in L_\alpha$. Since $[L_\alpha, L_{-\alpha}] = \operatorname{span}\{h_\alpha\}$, there exists a unique $Y_\alpha \in L_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = h_\alpha$. If $\{H_i\}$ is a basis of $\mathcal{H}$, then the set

$$\{\, X_\alpha, Y_\alpha \mid \alpha \succ 0 \,\} \cup \{\, H_i \,\}$$

is a basis of L consisting of Cartan elements and root vectors.

One further bit of notation is required. We let $\mathfrak{N}^-$ denote the subalgebra of L generated by the root spaces corresponding to negative roots (i.e. $\mathfrak{N}^- = \sum_{\alpha \prec 0} L_\alpha$). Similarly, $\mathfrak{N}^+ = \sum_{\alpha \succ 0} L_\alpha$. Also, we let $\mathcal{B} = \mathcal{H} + \mathfrak{N}^+$ denote the subalgebra of L generated by the Cartan elements and the root spaces corresponding to positive roots.

1.9: Theory of Weights

For later use, we develop some of the theoretical aspects of weights. Fix a root system Φ and simple base Δ. We define the **root lattice** Q to be the additive abelian subgroup of $(\mathbb{R}^l, +)$ given by $Q = \mathbb{Z}[\Phi] = \mathbb{Z}[\Delta]$, i.e. the set of all integer linear combinations of roots. Let $\{\, \omega_i \,\}$ be the basis of $\mathcal{H}$ which is dual to the basis $\{\, \frac{2\alpha_i}{(\alpha_i, \alpha_i)} \,\}$ with respect to the form on $\mathcal{H}^*$ (where $\alpha_i \in \Delta$). In other words,

$$\langle \omega_i, \alpha_j \rangle = \left(\omega_i, \frac{2\alpha_j}{(\alpha_j, \alpha_j)} \right) = \delta_{ij}.$$

Let $P = \mathbb{Z}[\{\, \omega_i \,\}]$ be the additive abelian subgroup of $(\mathbb{R}^l, +)$ consisting of all integer linear combinations of the basis elements $\{\, \omega_i \,\}$. We call P the **weight lattice** of

Φ and elements of P are called **weights**. Clearly $Q \subseteq P$. It is also clear that

$$P = \{\, \lambda \in \mathbb{R}^l \mid \langle \lambda, \alpha_i \rangle \in \mathbb{Z} \ \text{ for all } \alpha \in \Phi \,\}.$$

Let

$$P^+ = \{\, \lambda \in P \mid \langle \lambda, \alpha_i \rangle \geq 0 \ \text{ for all } \alpha_i \in \Delta \,\}.$$

Elements of P^+ are called **dominant weights** and the $\{\, \omega_i \,\}$ are called the **fundamental dominant weights** (relative to Δ). Note that if $\alpha_i = \sum c_j\, \omega_j$ then

$$\langle \alpha_i, \alpha_{j_0} \rangle = \sum c_j \, \langle \omega_j, \alpha_{j_0} \rangle = c_{j_0},$$

and hence $\alpha_i = \sum_{j=1}^{l} \langle \alpha_i, \alpha_j \rangle \, \omega_j$.

We define a partial ordering on P as follows: for $\lambda, \mu \in P$, $\lambda \succ \mu$ iff $\lambda - \mu \in Q^+$. Also, we define an important element of P, which we will use extensively, by $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

<u>**Proposition 1.40:**</u> $\delta = \sum_{i=1}^{l} \omega_i$.

<u>**Proof:**</u> Write our fixed simple base as $\Delta = \{\, \alpha_1, \ldots, \alpha_l \,\}$ and let $\sigma_i = \sigma_{\alpha_i} \in \mathcal{W}$ be the reflection in the plane perpendicular to α_i. Then clearly $\sigma_i(\alpha_i) = -\alpha_i$ and for $i \neq j$,

$$\sigma_i(\alpha_j) = \alpha_j - \langle \alpha_j, \alpha_i \rangle \, \alpha_i \in \Phi^+,$$

since $\langle \alpha_j, \alpha_i \rangle \leq 0$. Hence for any $\beta \in \Phi^+$, $\beta \neq \alpha_i$, we have $\sigma_i(\beta) \in \Phi^+$. Then

$$\begin{aligned} \sigma_i(\delta) &= \frac{1}{2} \sum_{\alpha \in \Phi^+} \sigma_i(\alpha) \\ &= \frac{1}{2} \sum_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \alpha_i}} \sigma_i(\alpha) + \frac{1}{2} \sigma_i(\alpha_i) \end{aligned}$$

$$= \frac{1}{2} \sum_{\substack{\alpha \in \Phi^+ \\ \alpha \neq \alpha_i}} \alpha - \frac{1}{2}\alpha_i$$

$$= \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha - \alpha_i = \delta - \alpha_i$$

But by definition, $\sigma_i(\delta) = \delta - \langle \delta, \alpha_i \rangle \alpha_i$ and hence $\langle \delta, \alpha_i \rangle = 1$. Since this holds for each i, $\delta = \sum_{i=1}^{l} \langle \delta, \alpha_i \rangle \omega_i = \sum_{i=1}^{l} \omega_i$. ∎

1.10: Universal Enveloping Algebras

Of particular importance in the study of representations of a Lie algebra L is the universal enveloping algebra of L. In this section we define this structure and discuss some important properties. Note that although L is assumed to be semisimple, the definitions and results here hold for arbitrary Lie algebras.

If V is a finite dimensional vector space over $\mathbb{C}$ we define the **tensor algebra** of V to be

$$\mathfrak{T}(V) := \sum_{i=0}^{\infty} \oplus T^i(V)$$

where $T^i(V) = V \otimes V \otimes \cdots \otimes V$ is the tensor product of i copies of V, and $T^0(V) = \mathbb{C}$. Multiplication in $\mathfrak{T}(V)$ is defined by juxtaposing with $\otimes$, i.e.

$$(v_1 \otimes \cdots \otimes v_i) \otimes (w_1 \otimes \cdots \otimes w_j) = v_1 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_j.$$

This makes $\mathfrak{T}(V)$ an associative $\mathbb{C}$-algebra with 1. The tensor algebra is also universal, in the following sense:

<u>Lemma 1.41:</u> *For any $\mathbb{C}$-linear map $\phi: V \to \mathfrak{A}$ into an associative $\mathbb{C}$-algebra with 1, there exists a unique $\mathbb{C}$-algebra homomorphism $\overline{\phi}: \mathfrak{T}(V) \to \mathfrak{A}$ such that $\overline{\phi}(1) = 1$ and $\overline{\phi} \downarrow_V = \phi$.*

Proof: Since $V = T^1(V)$ and since $T^1(V)$, along with the scalars, generate $\mathfrak{T}(V)$, then if we stipulate that $\overline{\phi}(1) = 1$, there exists a unique extension of the map $\phi\colon V \to \mathfrak{A}$ to a map $\overline{\phi}\colon \mathfrak{T}(V) \to \mathfrak{A}$ such that $\phi = \overline{\phi} \circ \imath$, where $\imath$ is the isomorphism $V \to T^1(V)$. ∎

If L is a Lie algebra over $\mathbb{C}$, then by a **universal enveloping algebra** of L we mean a pair $(\mathfrak{U}, \imath)$ where:

(1) $\mathfrak{U}$ is an associative $\mathbb{C}$-algebra with 1.

(2) $\imath\colon L \to \mathfrak{U}$ is a $\mathbb{C}$-linear map having the property that for all $x, y \in L$,

$$\imath\big([x,y]\big) = \imath(x)\,\imath(y) - \imath(y)\,\imath(x) \qquad ; \text{and}$$

(3) for every $\mathbb{C}$-linear map $\rho\colon L \to \mathfrak{A}$ into an associative $\mathbb{C}$-algebra $\mathfrak{A}$ with 1, such that for all $x, y \in L$,

$$\rho\big([x,y]\big) = \rho(x)\,\rho(y) - \rho(y)\,\rho(x),$$

there exists a unique $\mathbb{C}$-algebra homomorphism $\overline{\rho}\colon \mathfrak{U} \to \mathfrak{A}$ such that $\overline{\rho}(1) = 1$ and $\overline{\rho} \circ \imath = \rho$.

We note first that for a given L there is exactly one universal enveloping algebra of L.

Proposition 1.42: *If L is a Lie algebra then there exists a unique universal enveloping algebra of L.*

Proof: The uniqueness is easy to show. Suppose $(\mathfrak{U}_1, \imath_1)$ and $(\mathfrak{U}_2, \imath_2)$ are two universal enveloping algebras of L. Since $\imath_1\colon L \to \mathfrak{U}_1$, it has a unique extension to a $\mathbb{C}$-algebra homomorphism $\phi\colon \mathfrak{U}_2 \to \mathfrak{U}_1$. Similarly, $\imath_2$ has a unique extension to a $\mathbb{C}$-algebra homomorphism $\psi\colon \mathfrak{U}_1 \to \mathfrak{U}_2$. Again by definition there is a unique extension of $\imath_1$ to a $\mathbb{C}$-algebra homomorphism $\mathfrak{U}_1 \to \mathfrak{U}_1$. But both $\phi \circ \psi$ and $1_{\mathfrak{U}_1}$

satisfy this and hence by uniqueness, $\phi \circ \psi = 1_{\mathcal{U}_1}$. Similarly, $\psi \circ \phi = 1_{\mathcal{U}_2}$. Thus $\mathcal{U}_1 \cong \mathcal{U}_2$ as $\mathbb{C}$-algebras.

To show existence, we construct a universal enveloping algebra from the tensor algebra $\mathfrak{T}(L)$. Let $\mathfrak{J}$ be the two-sided ideal in $\mathfrak{T}(L)$ generated by

$$\{x \otimes y - y \otimes x - [x,y] \mid x, y \in L\}.$$

Set $\mathcal{U}(L) = \mathfrak{T}(L)/\mathfrak{J}$ and let π be the canonical homomorphism $\mathfrak{T}(L) \to \mathfrak{T}(L)/\mathfrak{J}$. Let $i = \pi\downarrow_L$. We claim that the pair $(\mathcal{U}(L), i)$ is a universal enveloping algebra of L. Since $\mathfrak{J} \subseteq \sum_{i=1}^{\infty} \oplus T^i(L)$, $\mathbb{C} = T^0(L) \subseteq \mathcal{U}(L)$, and so $\mathcal{U}(L)$ is an associative $\mathbb{C}$-algebra with 1. Also, $\mathfrak{J} = \operatorname{Ker} \pi$ and so i clearly satisfies property (2) of the definition. Now suppose that $\rho: L \to \mathfrak{A}$ is given as in the definition. By the universal property of $\mathfrak{T}(L)$, there is a unique map $\overline{\rho}: \mathfrak{T}(L) \to \mathfrak{A}$ which extends ρ and sends 1 to 1. Since $\rho([x,y]) = \rho(x)\,\rho(y) - \rho(y)\,\rho(x)$ it is clear that all elements $x \otimes y - y \otimes x - [x,y]$ lie in $\operatorname{Ker}\overline{\rho}$. Hence $\overline{\rho}$ induces a homomorphism $\overline{\overline{\rho}}: \mathcal{U}(L) \to \mathfrak{A}$ such that $\overline{\overline{\rho}} \circ i = \rho$. Finally, because 1 and $\operatorname{Im} i$ generate $\mathcal{U}(L)$, the map $\overline{\overline{\rho}}$ is unique. ∎

Thus we speak of *the* universal enveloping algebra of L and denote it by $\mathcal{U} = \mathcal{U}(L)$.

The following result is used a great deal in representation theory; however, as its proof involves some digression, we refer the reader to Humphreys.

Theorem 1.43 (Poincaré-Birkhoff-Witt): *Let $(x_1, x_2, \ldots, x_n)$ be an ordered basis of L. Then 1, along with the elements of $\mathcal{U}(L)$ of the form $x_{j_1} \cdots x_{j_m} = \pi(x_{j_1} \otimes \cdots \otimes x_{j_m})$ (where $m \in \mathbb{Z}^+$ and $1 \leq j_1 \leq \cdots \leq j_m \leq n$) form a basis for $\mathcal{U}(L)$.*

Proof: (see [Hu 1] §17.3–4). ∎

A basis of $\mathfrak{U}(L)$ of the type constructed in the theorem is called a **Poincaré-Birkhoff-Witt basis**, or more simply a **P-B-W basis**. Note the dependency of such a basis on the basis chosen for L.

One additional property of the universal enveloping algebra will be needed in what follows. The above theorem implies that there is a bijective correspondence between L-modules and $\mathfrak{U}(L)$-modules. In particular, any L-module V can be viewed as a module for $\mathfrak{U}(L)$ and vice-versa.

Chapter 2

Representation Theory of Semisimple Lie Algebras

In this chapter we develop the tools of representation theory which we will need in Chapter 3. Throughout this chapter we assume that L is a semisimple Lie algebra over $\mathbb{C}$.

2.1 Weight Space Decompositions of L-Modules

We first must generalize the concept of weight given in Chapter 1. If $\mathcal{H}$ is a Cartan subalgebra of L and $\lambda \in \mathcal{H}^*$ then λ is said to be a **weight function** (or more simply a **weight**) of the L-module V iff

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)\, v \text{ for all } h \in \mathcal{H}\} \neq (0).$$

We say that an L-module V **admits a weight space decomposition** with respect to a Cartan subalgebra $\mathcal{H}$ of L iff

$$V = \sum_{\lambda \in \mathcal{H}^*} \oplus V_\lambda.$$

<u>**Proposition 2.1:**</u> *Let V be an arbitrary L-module. Then*

(a) $L_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$ for all $\alpha \in \Phi$, $\lambda \in \mathcal{H}^*$;

(b) If $V' = \sum_{\lambda \in \mathcal{H}^*} V_\lambda$ then the sum is direct and $V' \leq V$;

(c) If $\dim V < +\infty$ *then* $V' = V$ *for* <u>*any*</u> *Cartan subalgebra* $\mathcal{H} \leq L$*;*

(d) If V *is simple and* $V_\lambda \neq (0)$ *for some* $\lambda \in \mathcal{H}^*$ *then* $V' = V$*.*

<u>**Proof:**</u> (a) Let $v \in V_\lambda$, $x \in L_\alpha$, and $h \in \mathcal{H}$. Then

$$\begin{aligned} h \cdot (x \cdot v) &= x \cdot (h \cdot v) + [h, x] \cdot v \\ &= \lambda(h)\, x \cdot v + \alpha(h)\, x \cdot v \\ &= (\lambda + \alpha)(h)\, x \cdot v. \end{aligned}$$

(b) Suppose the sum is not direct. Then there exists an expression of the form $\sum_{i=1}^{n} v_i = 0$ where $0 \neq v_i \in V_{\lambda_i}$ with the λ_i's distinct. Without loss of generality, we may assume that n is minimal in this expression. Since the λ_i's are distinct, in particular $\lambda_1 \neq \lambda_2$, and so we can find $h \in \mathcal{H}$ for which $\lambda_1(h) \neq \lambda_2(h)$. Now we have $\lambda_1(h) \sum v_i = 0$ and $h \cdot \sum v_i = \sum \lambda_i(h) v_i = 0$. Subtracting, we have that $\sum_{i=2}^{n} (\lambda_1 - \lambda_i)(h) v_i = 0$. But this last sum is nontrivial and has fewer than n terms, contrary to the minimality of n. Thus V' is a direct sum of weight spaces. Using (a) we have that $V' \leq V$.

(c) Since $\dim V < +\infty$, V is completely reducible by Weyl's Theorem. Without loss of generality, then, we may assume that V is simple. Let $\mathcal{H}$ be any Cartan subalgebra and let $0 \neq h \in \mathcal{H}$. Then h has an eigenvector in V and since $\mathcal{H}$ is abelian there exists a common eigenvector for all $h \in \mathcal{H}$. Hence there is a $\lambda \in \mathcal{H}^*$ such that $V_\lambda \neq (0)$. Using part (b), we have $V' \neq (0)$ and $V' \leq V$. By simplicity, $V' = V$, as required.

(d) follows as in (c). ∎

To study finite-dimensional L-modules, it suffices, by Weyl's Theorem and part (c) of Proposition 2.1, to consider only simple weighted modules.

Proposition 2.2: *If V is an L-module which admits a weight space decomposition with respect to some fixed Cartan subalgebra $\mathcal{H} \leq L$ and $W \leq V$, then*

$$W = \sum_{\lambda \in \mathcal{H}^*} \oplus (W \cap V_\lambda).$$

Proof: We prove in fact that if $\sum_{i=1}^n v_i \in W$ where $v_i \in V_{\lambda_i}$ with λ_i's distinct, then $v_i \in W$ for all i.

Suppose instead that there is a sum of the form $\sum_{i=1}^n v_i \in W$ with $v_i \in V_{\lambda_i}$ for distinct λ_i, and such that for some i, $v_i \notin W$. Without loss of generality, we can choose n minimal. If $v_1 \notin W$ and $v_i \in W$ for $i = 2, \ldots, n$, then as in the proof of part (b) of Proposition 2.1, we can choose $h \in \mathcal{H}$ for which $\lambda_1(h) \neq \lambda_n(h)$. Now we have $\lambda_n(h) \sum v_i = 0$ and $h \cdot \sum v_i = \sum \lambda_i(h) v_i = 0$. Subtracting, we have that $\sum_{i=1}^{n-1} (\lambda_n - \lambda_i)(h) v_i = 0$. But this last sum is nontrivial and has fewer than n terms, contrary to the minimality of n. Similarly, if $v_i \notin W$ for $i \neq 1$, then we can choose $h \in \mathcal{H}$ for which $\lambda_1(h) \neq \lambda_2(h)$. Then $\lambda_1(h) \sum v_i = 0$ and $h \cdot \sum v_i = \sum \lambda_i(h) v_i = 0$. Subtracting, we have that $\sum_{i=2}^{n} (\lambda_1 - \lambda_i)(h) v_i = 0$. Again this last sum is nontrivial and has fewer than n terms, contradicting the minimality of n. This proves our claim. ∎

The above propositions can now be used to give several equivalent conditions for a simple L-module to have a weight space decomposition with respect to a given Cartan subalgebra.

Theorem 2.3: *Let V be a simple L-module, $\mathcal{H}$ a Cartan subalgebra of L. The following are equivalent.*

(1) V has a weight space decomposition with respect to $\mathcal{H}$;

(2) For all $v \in V$, $\dim \mathfrak{U}(\mathcal{H}) \cdot v < +\infty$;

(3) There exists nonzero $v \in V$ such that $\dim \mathfrak{U}(\mathcal{H}) \cdot v < +\infty$;

(4) For all $v \in V$ and all $h \in \mathcal{H}$, $\dim \mathfrak{U}(\mathbb{C} \cdot h) \cdot v < +\infty$;

(5) There exists $v \in V$ such that for all $h \in \mathcal{H}$, $\dim \mathfrak{U}(\mathbb{C} \cdot h) \cdot v < +\infty$.

Proof: First note that we clearly have (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (2) $\Rightarrow$ (4), and (3) $\Rightarrow$ (5) since in each case we are going from a stronger statement to a weaker one.

(1) $\Rightarrow$ (2): This follows immediately from the fact that every $v \in V$ can be written as a finite linear combination of weight vectors.

(5) $\Rightarrow$ (3): For each $h \in \mathcal{H}$ there exists k_h such that

$$h^{k_h} v = \text{ linear comb. of } h^l v \quad \text{for } l < k_h$$

Take a basis $\{h_1, \ldots, h_r\}$ of $\mathcal{H}$ and then find $k_1, \ldots, k_r$ as above. Then the space spanned by the set

$$\left\{ h_1^{l_1} \cdots h_r^{l_r} v \mid l_i \leq \max\{k_1, \ldots, k_r\} \right\}$$

is invariant under $\mathfrak{U}(\mathcal{H})$ since $\mathcal{H}$ is commutative.

(3) $\Rightarrow$ (1): Since (3) holds, there exists a nonzero weight space with respect to $\mathcal{H}$. But V is simple and so by part (d) of Proposition 2.1, V has a weight space decomposition with respect to $\mathcal{H}$. ∎

Now fix a Cartan subalgebra $\mathcal{H}$ of L and consider the adjoint action of L on $\mathfrak{U} = \mathfrak{U}(L)$. We define a weight space decomposition of $\mathfrak{U}$ as

$$\mathfrak{U} = \sum_{\xi \in \mathcal{H}^*} \oplus \mathfrak{U}_\xi$$

2. Representation Theory of Semisimple Lie Algebras

Recall that in Section 1.8 we constructed a basis for L of the form

$$\{X_\alpha, Y_\alpha \mid \alpha \succ 0\} \cup \{H_i\},$$

where $\{H_i\}$ is a basis of $\mathcal{H}$, and for $\alpha \succ 0$, $X_\alpha \in L_\alpha$ and $Y_\alpha \in L_{-\alpha}$. Fix such a basis for L. Then a basis for $\mathfrak{U}_\xi$ is given by

$$\left\{ \prod_{\mu \in \Phi^+} X_\mu^{e_\mu} \prod_{\alpha_i \in \Delta} H_{\alpha_i}^{k_i} \prod_{\mu \in \Phi^+} Y_\mu^{f_\mu} \middle| e_\mu, k_i, f_\mu \geq 0 \text{ and } \sum_{\mu \in \Phi^+} (e_\mu - f_\mu)\mu = \xi \right\}$$

where Δ is a simple base for the root system Φ and Φ^+ is the set of positive roots with respect to Δ.

Observe that $\mathfrak{U}_\xi \mathfrak{U}_\eta \subseteq \mathfrak{U}_{\xi+\eta}$. In particular, then, we have that $\mathfrak{U}_0$ is a subalgebra of $\mathfrak{U}$ and $\mathfrak{U}_\xi$ is a $\mathfrak{U}_0$-module.

<u>Proposition 2.4:</u> *For any ξ in the root lattice of L, $\mathfrak{U}_\xi$ is a finitely-generated $\mathfrak{U}_0$-module. In particular, $\mathfrak{U}_0$ is a finitely-generated subalgebra of $\mathfrak{U}$.*

<u>Proof:</u> Let $\{E_1, \ldots, E_m\}$ be an ordered basis of L. Then $\mathfrak{U}$ has a basis $\mathcal{B} = \{E_1^{f_1} \cdots E_m^{f_m} \mid f_i \geq 0\}$. If $u \in \mathcal{B}$, we denote by $\mathrm{ex}_i(u)$ the ith exponent of u (ie. $\mathrm{ex}_i(u) = f_i$). We will call an element $u \in \mathfrak{U}_\xi \cap \mathcal{B}$ **minimal** iff for any other $u' \in \mathfrak{U}_\xi \cap \mathcal{B}$ with

$$\mathrm{ex}_i(u') \leq \mathrm{ex}_i(u) \quad \text{for all } i = 1, 2, \ldots, m$$

we have that necessarily $u' = u$.

We claim that there are only finitely many minimal elements in $\mathfrak{U}_\xi \cap \mathcal{B}$ and that they generate $\mathfrak{U}_\xi$ as a $\mathfrak{U}_0$-module.

In order to prove the first statement, we need to find a constant K such that for all minimal $u \in \mathfrak{U}_\xi \cap \mathcal{B}$, we have $\mathrm{ex}_i(u) \leq K$ for all $i = 1, 2, \ldots, m$.

First take any minimal element $u \in \mathfrak{U}_\xi \cap \mathfrak{B}$ and let $K_1 = \max\{\, \mathrm{ex}_i(u) | i = 1, \ldots, m\}$. We show that for every minimal $u' \in \mathfrak{U}_\xi \cap \mathfrak{B}$, there is at least one j such that $\mathrm{ex}_j(u') \leq K_1$. Indeed, suppose this is false. Then there exists a minimal $u' \in \mathfrak{U}_\xi \cap \mathfrak{B}$ such that $\mathrm{ex}_i(u') > K_1$ for all i. But for each i, $\mathrm{ex}_i(u) \leq K_1 < \mathrm{ex}_i(u')$ which contradicts the minimality of u'.

Assume inductively that there exists K_r such that for every minimal $u \in \mathfrak{U}_\xi \cap \mathfrak{B}$,

$$|\{ j \mid \mathrm{ex}_j(u) \leq K_r \,\}| \geq r \,.$$

Also, let

$$I^{(r)} = \{\, (n_1, \ldots, n_r) \mid 1 \leq n_1 < \cdots < n_r \leq m \}$$

and

$$J^{(r)} = \{\, (m_1, \ldots, m_r) \mid 0 \leq m_i \leq K_r \ \text{ and } i = 1, \ldots, r \,\}$$

Then since $I^{(r)}$ and $J^{(r)}$ are bounded, we have that $\left| I^{(r)} \times J^{(r)} \right| < +\infty$.

Now for each $(\underline{n}, \underline{m}) \in I^{(r)} \times J^{(r)}$ we let

$$P_{\underline{n},\underline{m}} = \{ \text{ minimal } u \in \mathfrak{U}_\xi \cap \mathfrak{B} \mid \ \mathrm{ex}_{n_i}(u) = m_i \,;\, i = 1, \ldots, r \}$$

If $P_{\underline{n},\underline{m}} = \emptyset$ we let $K_{\underline{n},\underline{m}} = 0$. On the other hand, if $P_{\underline{n},\underline{m}} \neq \emptyset$ we select $u_{\underline{n},\underline{m}} \in P_{\underline{n},\underline{m}}$ and define

$$K_{\underline{n},\underline{m}} = \max \{ \{\, \mathrm{ex}_i(u_{\underline{n},\underline{m}}) \mid i = 1, \ldots, m \} \cup \{ K_r \} \}$$

Now define

$$K_{r+1} = \max \left\{ K_{\underline{n},\underline{m}} \mid (\underline{n}, \underline{m}) \in I^{(r)} \times J^{(r)} \right\}$$

Assume there exists a minimal $u' \in \mathfrak{U}_\xi \cap \mathfrak{B}$ such that $\mathrm{ex}_i(u') \leq K_{r+1}$ for *exactly* r indices $(n_1, \ldots, n_r) = \underline{n}$. Let $m_i = \ \mathrm{ex}_{n_i}(u')$ and let $\underline{m} = (m_1, \ldots, m_r)$. Note that $m_i \leq K_r$ for $i = 1, 2, \ldots, r$ by our inductive hypothesis.

Consider $u_{\underline{n},\underline{m}}$. Note that

$$\mathrm{ex}_i(u_{\underline{n},\underline{m}}) = \begin{cases} m_j = \ \mathrm{ex}_i(u') \text{ if } i = n_j \\ \leq K_{r+1} < \ \mathrm{ex}_i(u') \text{ if } i \notin \{n_1, \ldots, n_r\} \end{cases}$$

But this contradicts the minimality of u' and hence there are only finitely many minimal elements in $\mathfrak{U}_\xi \cap \mathfrak{B}$.

To prove that the minimal elements generate $\mathfrak{U}_\xi$ as a $\mathfrak{U}_0$-module, note first that any commutator product of basis elements of L can be written as a linear combination of basis elements. Notice that the elements of $\mathfrak{U}_\xi \cap \mathfrak{B}$ form a basis for $\mathfrak{U}_\xi$. Let $u = E_1^{f_1} \cdots E_m^{f_m} \in \mathfrak{U}_\xi \cap \mathfrak{B}$ with $f_i \geq 0$. Define the degree of u to be $\deg u = \sum_i f_i$.

We induct on the degree of u. If u is minimal, then we are done. So suppose otherwise. Then there must be minimal $u' = E_1^{g_1} \cdots E_m^{g_m} \in \mathfrak{U}_\xi \cap \mathfrak{B}$ with $f_i \geq g_i \geq 0$ for each i and with at least one i such that $f_i > g_i$. Then $u = u'u'' + z$ where u'' is an element in $\mathfrak{U}_\xi$ with $\deg u'' < \deg u$, and z is a linear combination of terms which contain a commutator product of two basis elements of L. In particular, since each such commutator product can be expressed as a linear combination of basis elements of L, z can be rewritten as a linear combination of terms of degree $< \deg u$. By the inductive hypothesis, each term in this expression for z can be generated by minimal elements, and hence so can z. Similarly, u'' can be generated by minimal elements. Thus u can be so generated. This says that a basis of $\mathfrak{U}_\xi$ can be so generated, and hence all of $\mathfrak{U}_\xi$ can be. ∎

Using the above results, we may now prove two additional results regarding weight space decompositions.

<u>Theorem 2.5:</u> *If V is a simple L-module having a finite-dimensional weight space*

with respect to a Cartan subalgebra $\mathcal{H} \leq L$, then V has a weight space decomposition with respect to $\mathcal{H}$ in which all weight spaces are finite-dimensional.

Proof: We are given the existence of $\lambda \in \mathcal{H}^*$ such that $V_\lambda \neq (0)$ and $\dim V_\lambda < +\infty$. By part (d) of Proposition 2.1, we have

$$V = \sum_{\mu \in \mathcal{H}^*} \oplus V_\mu$$

Take any nonzero $v_0 \in V_\lambda$. Since V is simple, $\mathcal{U} \cdot v_0 = V$. Therefore, we must have

$$V = \mathcal{U} \cdot v_0 = \sum_{\substack{\xi \in \text{root} \\ \text{lattice}}} \oplus \mathcal{U}_\xi \cdot v_0$$

If $\mathcal{U}_\xi \cdot v_0 \neq (0)$ then $\mathcal{U}_\xi \cdot v_0 \subseteq V_{\lambda+\xi}$. Since $\mathcal{U} \cdot v_0 = V$ we must in fact have $\mathcal{U}_\xi \cdot v_0 = V_{\lambda+\xi}$. In particular this implies that $V_\lambda = \mathcal{U}_0 \cdot v_0$.

Since V_λ is finite-dimensional, there exists a basis $\{ e_i \cdot v_0 \mid i = 1, 2, \ldots, k \}$ where $e_i \in \mathcal{U}_0$. Consider $V_\mu = \mathcal{U}_{\mu-\lambda} \cdot v_0$. By Proposition 2.4, $\mathcal{U}_{\mu-\lambda}$ is finitely generated as a $\mathcal{U}_0$-module and hence for any $u \in \mathcal{U}_{\mu-\lambda}$ there exist $u_1, \ldots, u_l \in \mathcal{U}_{\mu-\lambda}$ such that $u = \sum_{i=1}^{l} u_i c_i$ where $c_i \in \mathcal{U}_0$. In addition, since $\mathcal{U}_0$ is finitely generated, we have that $c_i \cdot v_0 = \sum_{j=1}^{k} b_{ij} e_j v_0$. Thus for any $u \in \mathcal{U}_{\mu-\lambda}$,

$$\begin{aligned} u \cdot v_0 &= \sum_{i=1}^{l} u_i (c_i \cdot v_0) \\ &= \sum_{i=1}^{l} \sum_{j=1}^{k} b_{ij} u_i e_j \cdot v_0 \end{aligned}$$

But this says that $\mathcal{U}_{\mu-\lambda} \cdot v_0$ is spanned by

$$\left\{ u_i e_j \cdot v_0 \;\middle|\; \begin{matrix} i = 1, \ldots, l \\ j = 1, \ldots, k \end{matrix} \right\}. \blacksquare$$

Theorem 2.6: *(a) If V is a simple L-module with a weight space decomposition with respect to a Cartan subalgebra $\mathcal{H} \leq L$, then each weight space V_λ is a simple $\mathcal{U}_0$-module (recall that $\mathcal{U}_0$ depends on our choice of $\mathcal{H}$).*

(b) If W is a simple finite-dimensional $\mathcal{U}_0$-module then (up to equivalence) there exists a unique L-module having a weight space decomposition with respect to $\mathcal{H}$ such that one of its weight spaces is isomorphic to W as a $\mathcal{U}_0$-module.

Proof: (a) Consider the weight space V_λ, for a fixed λ. Recall that by part (a) of Proposition 2.1, for any $\alpha \in \Phi$, $L_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$. This implies that $\mathcal{U}_\xi V_\lambda \subseteq V_{\lambda+\xi}$ and hence $\mathcal{U}_0 V_\lambda \subseteq V_\lambda$. Also, we have that for $\xi \neq 0$, $\mathcal{U}_\xi V_\lambda \cap V_\lambda = \emptyset$. By simplicity of V, we must have $\mathcal{U}_0 V_\lambda = V_\lambda$. Moreover, if $v_0 \in V_\lambda$ with $v_0 \neq 0$, then since $V = \mathcal{U} \cdot v_0 = \sum \mathcal{U}_\xi \cdot v_0$, the simplicity of V again implies that $\mathcal{U}_0 \cdot v_0 = V_\lambda$. Therefore, V_λ is a simple $\mathcal{U}_0$-module.

(b) Let W be as given. Note first that $\mathcal{H}$ is contained in the center of $\mathcal{U}_0$ and hence any $h \in \mathcal{H}$ acts as a scalar on W, i.e. there exists $\lambda \in \mathcal{H}^*$ such that $h \cdot w = \lambda(h)w$ for all $w \in W$.

Fix $w_0 \in W$ and consider $\operatorname{Ann} w_0 = \{u \in \mathcal{U}_0 \,|\, u \cdot w_0 = 0\}$. Then $\operatorname{Ann} w_0$ is a left ideal in $\mathcal{U}_0$, since if $u \in \mathcal{U}_0$ and $u' \in \operatorname{Ann} w_0$, then $(u\,u') \cdot w_0 = u\,(u' \cdot w_0) = 0$. Moreover, $\operatorname{Ann} w_0$ is actually a maximal left ideal in $\mathcal{U}_0$. To see this, suppose that $\mathcal{J}$ is a left ideal of $\mathcal{U}_0$ such that $\operatorname{Ann} w_0 \subsetneq \mathcal{J}$ and $z \in (\mathcal{J} \setminus \operatorname{Ann} w_0)$. Then $z \cdot w_0 \neq 0$ and so by the simplicity of W, there must exist $z' \in \mathcal{U}_0$ such that $(z'\,z) \cdot w_0 = w_0$. Hence $(z'\,z - 1) \cdot w_0 = 0$ and so $(z'\,z - 1) \in \operatorname{Ann} w_0 \subseteq \mathcal{J}$. But $z'\,z \in \mathcal{J}$, which implies that $1 \in \mathcal{J}$ and so $\mathcal{J} = \mathcal{U}_0$. Thus $\operatorname{Ann} w_0$ is indeed a maximal left ideal in $\mathcal{U}_0$.

In fact, we have that $W \cong \mathcal{U}_0/\operatorname{Ann} w_0$ as a $\mathcal{U}_0$-module. Indeed, consider the map $\phi\colon \mathcal{U}_0/\operatorname{Ann} w_0 \to W$ given by $u + \operatorname{Ann} w_0 \mapsto u \cdot w_0$. If $u + \operatorname{Ann} w_0 = u' + \operatorname{Ann} w_0$, then $(u - u') + \operatorname{Ann} w_0 = 0 + \operatorname{Ann} w_0$ and hence $(u - u') \cdot w_0 = 0 \cdot w_0 = 0$. Thus

$u \cdot w_0 = u' \cdot w_0$ and so ϕ is well-defined.

To see that ϕ is a $\mathfrak{U}_0$-module homomorphism, let $c \in \mathbb{C}$, $u, u' \in \mathfrak{U}_0$. Then

$$\begin{aligned}\phi\big(c(u + \operatorname{Ann} w_0) + (u' + \operatorname{Ann} w_0)\big) &= \phi\big((cu + u') + \operatorname{Ann} w_0\big) \\ &= (cu + u') \cdot w_0 = c(u \cdot w_0) + u' \cdot w_0 \\ &= c\,\phi(u + \operatorname{Ann} w_0) + \phi(u' + \operatorname{Ann} w_0)\end{aligned}$$

and

$$\begin{aligned}\phi\big(u \cdot (u' + \operatorname{Ann} w_0)\big) &= \phi\big((u\,u') + \operatorname{Ann} w_0\big) \\ &= (u \cdot u') \cdot w_0 = u \cdot (u' \cdot w_0) \\ &= u \cdot \phi(u' + \operatorname{Ann} w_0)\end{aligned}$$

The fact that ϕ is bijective is also easy to see. Indeed, if $\phi(u + \operatorname{Ann} w_0) = \phi(u' + \operatorname{Ann} w_0)$, i.e. $u \cdot w_0 = u' \cdot w_0$, then $(u - u') \cdot w_0 = 0$ and so $u - u' \in \operatorname{Ann} w_0$. Thus $u + \operatorname{Ann} w_0 = u' + \operatorname{Ann} w_0$ and so ϕ is 1-1. Also, since W is simple, $\mathfrak{U}_0 \cdot w_0 = W$, and hence if $w \in W$ there exists $u \in \mathfrak{U}_0$ such that $u \cdot w_0 = w$. Thus $\phi(u + \operatorname{Ann} w_0) = w$ and so ϕ is onto. Therefore we indeed have $W \cong \mathfrak{U}_0 / \operatorname{Ann} w_0$.

Now let $\mathfrak{J} = \mathfrak{U} \cdot (\operatorname{Ann} w_0)$. Reasoning as above, we have that $\mathfrak{J}$ is a proper left ideal in $\mathfrak{U}$. We claim that since $\operatorname{Ann} w_0 \subseteq \mathfrak{J}$, we may write $\mathfrak{J} = \sum_\xi (\mathfrak{J} \cap \mathfrak{U}_\xi)$. To see this, first note that since $\operatorname{Ann} w_0 \subseteq \mathfrak{J}$, we may express any $x \in \mathfrak{J}$ as a sum $x = \sum_{i=1}^m x_i$ where $x_i \in \mathfrak{U}_{\xi_i}$ and the ξ_i's are distinct. Choose m minimal such that $x = \sum_{i=1}^m x_i \in \mathfrak{J}$, and yet $x_i \notin \mathfrak{J}$ for $i = 1, \ldots, m$. Let $h \in \mathcal{H}$ be such that for $i \neq j$, $\xi_i(h) \neq \xi_j(h)$. Then

$$\xi_1(h)x - h \cdot x = \sum_{i=2}^m [\xi_1(h) - \xi_i(h)]x_i.$$

Clearly $\xi_1(h)x \in \mathfrak{J}$ and since $\mathfrak{J}$ is a left ideal in $\mathfrak{U}$, $h \cdot x \in \mathfrak{J}$. Therefore, $\sum_{i=2}^{m}[\xi_1(h) - \xi_i(h)]x_i \in \mathfrak{J}$, contrary to the minimality of m. Thus $x_i \in \mathfrak{J}$ for all i and so $\mathfrak{J} = \sum_\xi (\mathfrak{J} \cap \mathfrak{U}_\xi)$. In addition, it is clear that, since $\operatorname{Ann} w_0$ is a left ideal in $\mathfrak{U}_0$, $\mathfrak{J} \cap \mathfrak{U}_0 = \operatorname{Ann} w_0$.

If $V'/\mathfrak{J}$ is a proper submodule of $\mathfrak{U}/\mathfrak{J}$, then necessarily $1+\mathfrak{J} \notin V'/\mathfrak{J}$. In particular, $1 \notin V'$ and so, using the notation of Section 1.10, $V' \subseteq \left(\sum_{i=1}^{\infty} \oplus T^i(L)\right)/\mathfrak{I}$. Thus if $V'/\mathfrak{J}$ and $V''/\mathfrak{J}$ are proper submodules of $\mathfrak{U}/\mathfrak{J}$, then $V', V'' \subseteq \left(\sum_{i=1}^{\infty} \oplus T^i(L)\right)/\mathfrak{I}$, and so $V' + V'' \subseteq \left(\sum_{i=1}^{\infty} \oplus T^i(L)\right)/\mathfrak{I}$. This means that $1 \notin V' + V''$ and hence $1+\mathfrak{J} \notin V'/\mathfrak{J}+V''/\mathfrak{J}$, i.e. $V'/\mathfrak{J}+V''/\mathfrak{J}$ is a proper submodule of $\mathfrak{U}/\mathfrak{J}$. Therefore there exists a unique maximal submodule $M/\mathfrak{J}$ of $\mathfrak{U}/\mathfrak{J}$. Let $V = \mathfrak{U}/M \cong (\mathfrak{U}/\mathfrak{J})/(M/\mathfrak{J})$. We claim that V is the module we are seeking. Since M is a maximal left ideal in $\mathfrak{U}$, clearly V is a simple $\mathfrak{U}_0$-module. Hence

$$V = \mathfrak{U}/M = \sum_\xi \mathfrak{U}_\xi/(\mathfrak{U}_\xi \cap M)$$

and V has a weight space decomposition with respect to $\mathcal{H}$. In fact, we have that $\mathfrak{U}_\xi/(\mathfrak{U}_\xi \cap M)$ is the $\lambda + \xi$-weight space. To see this, let $h \in \mathcal{H}$ and $u \in \mathfrak{U}_\xi$. Then since M contains $\operatorname{Ann} w_0$ and $h - \lambda(h) \cdot 1 \in \operatorname{Ann} w_0$, we have that

$$\begin{aligned} h\,(u+M) &= [h,u] + u\,h + M \\ &= \xi(h)\,u + u\,(h+M) \\ &= \xi(h)\,u + \lambda(h)\,u + M \\ &= (\lambda+\xi)(h)\,(u+M). \end{aligned}$$

Finally,

$$\frac{\mathfrak{U}_0 + M}{M} \cong \frac{\mathfrak{U}_0}{(\mathfrak{U}_0 \cap M)} = \frac{\mathfrak{U}_0}{\operatorname{Ann} w_0} \cong W$$

as $\mathfrak{U}_0$-modules. ∎

2.2 Maximal Vectors and Standard Cyclic Modules

In this subsection, we fix a Cartan subalgebra $\mathcal{H}$ and a corresponding root system Φ. Fix a simple base Δ of Φ, and let Φ^+ denote the set of positive roots with respect to the partial order defined by Δ (similarly for Φ^-).

In section 1.6 we saw that any finite dimensional $sl(2,\mathbb{C})$-module has at least one maximal vector. In general, a Lie algebra L will have more than one positive root and hence we must generalize the concept of maximal vector. It turns out that for finite dimensional modules, maximal vectors exist; however, an infinite dimensional L-module need not contain a maximal vector.

A vector v in an L-module V is said to be a **maximal vector** of weight $\lambda \in \mathcal{H}^*$ iff $v \in V_\lambda$ and for $\mu \in \Phi^+$, $L_\mu\, v = 0$. We call an L-module V **standard cyclic** (or alternately a **highest weight** L-module) of highest weight λ iff $V = \mathfrak{U}v^+$ where v^+ is a maximal vector of weight λ. In this case, λ is referred to as the **highest weight** of V. For a standard cyclic module $V = V(\lambda)$ of highest weight λ, we let $\Pi(V(\lambda)) = \Pi(\lambda)$ denote the set of weight functions of V.

Standard cyclic modules have some interesting general properties.

<u>**Proposition 2.7:**</u> *Let V be a standard cyclic L-module, v^+ a maximal vector in V of highest weight λ. Then*

(a) $V = span\{\, Y_1^{f_1} \cdots Y_m^{f_1} \mid f_i \geq 0, Y_i \in L_{\mu_i} \}$ where $\Phi^- = \{\mu_1, \ldots, \mu_m\}$;

(b) V admits a weight space decomposition as does any submodule W of V;

(c)

$$\Pi(\lambda) \subseteq \{\, \lambda - \sum_{i=1}^{\ell} k_i\, \alpha_i \mid k_i \in \mathbb{Z},\ k_i \geq 0 \,\}; \tag{2.8}$$

(d) $\dim V_\lambda = 1$ and for any $\mu \in \Pi(\lambda)$, $\dim V_\mu < \infty$;

(e) V is an indecomposable L-module. Moreover, V contains a unique maximal

submodule and hence has a unique irreducible quotient;

(f) Every nonzero homomorphic image of V is again standard cyclic of highest weight λ.

<u>**Proof:**</u> (a) Let $\Phi^- = \{\mu_1, \ldots, \mu_m\}$ be an ordering of the negative roots with respect to Δ. Choose a P-B-W basis of $\mathfrak{U}$ so that the positive root vectors are to the right, followed by the Cartan elements, with the negative root vectors to the left, and such that the negative root vectors are ordered by the ordering on Φ^-. Let b be an arbitrary basis element. If b contains any positive root vectors, then by the definition of v^+, $b \cdot v^+ = 0$. On the other hand, if b does not contain any positive root vectors, then we may write $b = yh$ where y is a product of negative root vectors and h is a product of Cartan elements. Since v^+ is a weight vector, h acts on v^+ as a scalar and so $yh \cdot v^+ = c(y \cdot v^+)$. Hence

$$V = \operatorname{span}_{\mathbb{C}} \left\{ Y_1^{f_1} \cdots Y_m^{f_m} v^+ \;\middle|\; f_i \geq 0 \,;\, Y_i \in L_{\mu_i} \right\} .$$

(b) The first statement is a direct consequence of Theorem 2.3 and the second statement is a particular case of Proposition 2.2.

(c) By part (a) of Proposition 2.1, the vector

$$Y_{\beta_1}^{i_1} \cdots Y_{\beta_m}^{i_m} \cdot v^+ \tag{2.9}$$

has weight $\lambda - \sum_{j=1}^m i_j \beta_j$ with the i_j's being nonnegative integers. By rewriting each β_j as a nonnegative integer linear combination of the simple roots α_i, we get the result.

(d) Clearly (2.8) implies that the only vector of the form (2.9) which has weight λ is v^+ itself, and hence $\dim V_\lambda = 1$. Moreover, for any $\mu \in \Pi(\lambda)$, $\mu = \lambda - \sum_{i=1}^{l} k_i \alpha_i$ for some nonnegative integers k_i. Obviously there are only a finite number of vectors of the form (2.9) for which $\sum i_j \beta_j = \sum k_i \alpha_i$ and hence $\dim V_\mu < \infty$.

(e) If W is a proper submodule of V, then $W_\lambda = (0)$. Moreover, if W_1 and W_2 are proper submodules of V, then $(W_1 + W_2)_\lambda = (0)$ and so $W_1 + W_2$ is a proper submodule of V. Thus, in particular, V cannot be written as a sum of two proper submodules and hence is indecomposable. In addition, the above implies the existence of a unique proper submodule of V, namely the sum of all proper submodules. Therefore, as noted in the statement of the proposition, V has a unique irreducible quotient.

(f) If W is a nonzero homomorphic image of V (with ϕ the homomorphism), then for each $w \in W$, $w = \phi(v)$ for some $v \in V$. Hence $w = \phi(u \cdot v^+) = u \cdot \phi(v^+)$ for some $u \in \mathfrak{U}$. Thus $W = \mathfrak{U} \cdot \phi(v^+)$. In addition, we have that for $h \in \mathcal{H}$, $h \cdot \phi(v^+) = \phi(h \cdot v^+) = \lambda(h)\, \phi(v^+)$ and if $x \in L_\beta$ for $\beta \in \Phi^+$, $x \cdot \phi(v^+) = \phi(x \cdot v^+) = 0$. Hence $\phi(v^+)$ is a maximal vector in W and so W is standard cyclic of highestweight λ. ∎

We now wish to show that for each $\lambda \in \mathcal{H}^*$ there exists an irreducible standard cyclic module of highest weight λ, and that (up to equivalence) this module is unique. Here we follow the approach of Humphreys (see [Hu 1], pp. 109–10).

We first consider the uniqueness question.

<u>Theorem 2.10:</u> *Let V, W be standard cyclic modules of highest weight λ. If V and W are irreducible, then they are isomorphic.*

<u>Proof:</u> Consider the direct sum of V and W, $X = V \oplus W$, as an L-module and let v^+ and w^+ be maximal vectors (of weight λ) in V and W, respectively. Define

$x^+ = (v^+, w^+) \in X$. Then by the action of L defined on X, x^+ is clearly a maximal vector of weight λ.

Let Y be the L-submodule of X generated by x^+. Then Y is standard cyclic. If we now let $p\colon Y \to V$ and $p'\colon Y \to W$ be the maps induced by projecting X onto its first and second coordinates and restricting to Y. Clearly p and p' are L-module homomorphisms. Also, $p(x^+) = v^+$ and $p'(x^+) = w^+$, and hence by the irreducibility of V and W, we have that $\operatorname{Im} p = V$ and $\operatorname{Im} p' = W$.

Finally, V and W are irreducible quotients of the standard cyclic module Y and hence by part (e) of the previous proposition, $V \cong W$. ∎

The existence part is slightly more involved.

<u>**Theorem 2.11:**</u> *Let $\lambda \in \mathcal{H}^*$. Then there exists an irreducible standard cyclic L-module $V(\lambda)$ of highest weight λ.*

<u>**Proof:**</u> Let $\mathcal{B} = \mathcal{H} + \mathfrak{N}^+$, as in Section 1.8. Fix $\lambda \in \mathcal{H}^*$ and let D_λ be a one dimensional vector space, with basis vector v^+. We define an action of $\mathcal{B}$ on D_λ by setting

$$\Big(h + \sum_{\alpha \succ 0} x_\alpha\Big) \cdot v^+ = h \cdot v^+ = \lambda(h)\, v^+.$$

This clearly makes D_λ a $\mathcal{B}$-module, and hence we can consider D_λ as a $\mathfrak{U}(\mathcal{B})$-module. Thus we may form the tensor product module $Z(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(\mathcal{B})} D_\lambda$, which becomes a $\mathfrak{U}(L)$-module, under the natural left action of $\mathfrak{U}(L)$.

We claim that $Z(\lambda)$ is standard cyclic of highest weight λ. It is evident that $1 \otimes v^+$ generates $Z(\lambda)$. Also, the Poincaré-Birkhoff-Witt Theorem implies that $\mathfrak{U}(L)$ is a free $\mathfrak{U}(\mathcal{B})$-module, with free basis consisting of 1 along with the various monomials $Y_{\beta_1}^{i_1} \cdots Y_{\beta_m}^{i_m}$. Hence $1 \otimes v^+$ is nonzero and so $1 \otimes v^+$ is a maximal vector of weight λ.

Finally, by Proposition 2.7 (e), $Z(\lambda)$ contains a unique maximal submodule $Y(\lambda)$ and hence $V(\lambda) \cong Z(\lambda)/Y(\lambda)$ is an irreducible standard cyclic module of highest weight λ. ∎

The module $Z(\lambda)$ constructed in Theorem 2.11 is called the **Verma module** of highest weight λ.

We close this section with one further definition. For a standard cyclic L-module $V(\lambda)$ and $\mu \in \mathcal{H}^*$, we define the **multiplicity** of μ in $V(\lambda)$ to be $m_{V(\lambda)}(\mu) = \dim V(\lambda)_\mu$ if $\mu \in \Pi(\lambda)$ and $m_{V(\lambda)}(\mu) = 0$ if μ is not a weight of $V(\lambda)$. When no confusion can arise, we denote $m_{V(\lambda)}(\mu)$ by $m_\lambda(\mu)$. Also, if V is any finite dimensional module, $m_V(\mu)$ is similarly defined.

2.3 Criterion for Finite Dimension

Our goal in this section is to develop a criterion which will tell us for precisely which $\lambda \in \mathcal{H}^*$ the irreducible standard cyclic module $V(\lambda)$ is finite dimensional. If V is a finite dimensional irreducible L-module, then V has (up to scalar multiples) a unique maximal vector v^+ of highest weight λ, and the submodule of V generated by v^+ must, by irreducibility, be all of V. By the uniqueness theorem from Section 2.2, we must have $V \cong V(\lambda)$.

Theorem 2.12: *If V is a finite dimensional irreducible standard cyclic module of highest weight λ, then $\lambda(h_i)$ is a nonnegative integer (for $1 \leq i \leq l$).*

Proof: For each simple root α_i, let S_i denote the corresponding isomorphic copy of $sl(2,\mathbb{C})$ in L. Then $V(\lambda)$ is also a finite dimensional S_i-module, and moreover it is clear that a maximal vector for L is also a maximal vector for S_i. In particular, since there exists a maximal vector of weight λ, we have that the weight for the

Cartan subalgebra $\mathcal{H}_i$ of S_i is determined completely by $\lambda(h_i)$. But by Theorem 1.33, $\lambda(h_i)$ must be a nonnegative integer. ∎

The converse of this statement is not as easy to prove.

<u>Theorem 2.13:</u> *If $\lambda \in \mathcal{H}^*$ is a dominant integral weight, then the irreducible standard cyclic module of highest weight λ is finite dimensional. Moreover, the set of weights $\Pi(\lambda)$ is permuted by $\mathcal{W}$, with $\dim V_\mu = \dim V_{\sigma\mu}$ for all $\sigma \in \mathcal{W}$.*

<u>Proof:</u> Let v^+ be a maximal vector of weight λ in $V(\lambda)$. The proof proceeds in steps.

(1) We first claim that for each $1 \leq i \leq l$, V contains a nonzero finite dimensional S_i-module (where S_i is the copy of $sl(2,\mathbb{C})$ in L which corresponds to the simple root α_i). To see this, let $k = \langle \lambda, \alpha_i \rangle$. By Lemma 1.32, we must have that $v = Y_{\alpha_i}^{k+1} v^+$ is either 0 or a maximal vector for S_i of weight $\lambda - (k+1)\alpha_i$. Since for $j \neq i$, $\alpha_j - \alpha_i \notin \Phi$, we have $[Y_{\alpha_i}, X_{\alpha_j}] = 0$ and hence for $1 \leq i \leq l$, $X_{\alpha_i}(v) = 0$. But $V(\lambda)$ has no maximal vector of weight smaller than λ, and so $v = 0$. It follows then, by the actions of S_i given in Lemma 1.32 that $\mathrm{span}\{v^+, Y_{\alpha_i} v^+, \ldots, Y_{\alpha_i}^k v^+\}$ is the desired submodule.

(2) We now claim that the sum of all finite dimensional S_i-submodules of $V(\lambda)$ is $V(\lambda)$ itself. Indeed, it suffices to prove that this nonzero sum is stable under L, by the irreducibility of $V(\lambda)$. Begin with a nonzero finite dimensional S_i-submodule W of $V(\lambda)$ and let V' denote the span of all subspaces $X_\alpha(W)$ and $Y_\alpha(W)$ for $\alpha \in \Phi^+$. Then V' is clearly finite dimensional, as well as S_i-stable, and hence, since the X_α and Y_α generate L, V' is L-stable and so must be all of $V(\lambda)$.

(3) Recall that the reflections defined by the simple roots, i.e. $\sigma_i = \sigma_{\alpha_i}$, generate the group $\mathcal{W}$. We claim that if $\mu \in \Pi(\lambda)$, then so is $\sigma_i(\mu)$. To see this, notice first

that by step (2), $V(\lambda)_\mu + V(\lambda)_{\sigma_i(\mu)}$ lies in a finite dimensional S_i-submodule of $V(\lambda)$, and we may assume that this sum in fact lies in $\sum_{k\in\mathbb{Z}} V(\lambda)_{\mu+k\alpha_i}$. Theorem 1.33 then implies that μ and $\sigma_i(\mu) = \mu - \langle\mu,\alpha_i\rangle\,\alpha_i$ occur with the same multiplicity.

(4) Finally, we finish the proof. Since the σ_i generate $\mathcal{W}$, step (3) implies that the set $\Pi(\lambda)$ of weights of $V(\lambda)$ is stable under $\mathcal{W}$. Since there are only finitely many $\mu \in P^+$ with $\mu \prec \lambda$, the set $\Pi(\lambda)$ is bounded and hence, being discrete, $\Pi(\lambda)$ is finite. But each weight space of $V(\lambda)$ is finite dimensional, by Proposition 2.7 (d), and hence $V(\lambda)$ is finite dimensional, as required. ∎

2.4 Central Characters and the Harish-Chandra Theorem

Let $\mathfrak{Z} = Z(\mathfrak{U})$, i.e. the center of the universal enveloping algebra $\mathfrak{U}$ of L. An L-module V is said to **admit a central character** if there exists $v \in V$, $v \neq 0$, such that for all $z \in \mathfrak{Z}$, $z \cdot v = \chi(z)\,v$ where $\chi(z)$ is a scalar. Then the map $\chi\colon \mathfrak{Z} \to \mathbb{C}$ is called a **central character** of V. It is important to notice that this map is actually an algebra homomorphism. To see this note that linearity is clear, and moreover, for $z_1, z_2 \in \mathfrak{Z}$,

$$\chi(z_1\, z_2)\,v = (z_1\, z_2)\cdot v = \chi(z_2)\, z_1 \cdot v = \chi(z_1)\chi(z_2)\,v$$

and so χ preserves ordinary multiplication.

To consider a particular case, note that if $\lambda \in \mathcal{H}^*$ then the Verma module $Z(\lambda)$ admits exactly one central character. Indeed, if $v^+ \in Z(\lambda)$ is a maximal vector and $z \in \mathfrak{Z} \subseteq \mathfrak{U}_0$, $z \cdot v^+ = \chi(z)\,v^+$. Moreover, for any $v \in Z(\lambda)$, we have $v = u\,v^+$ for some $u \in \mathfrak{U}$, and hence

$$z \cdot v = z \cdot (u \cdot v^+) = u \cdot (z \cdot v^+) = \chi(z)\, u \cdot v^+ = \chi(z)\, v \tag{2.14}$$

We denote the central character of the Verma module $Z(\lambda)$ by χ_λ. Notice that if ϕ: $Z(\lambda) \to Y(\lambda)$ is a $\mathfrak{U}$-module homomorphism (i.e. $Y(\lambda)$ is a standard cyclic module of highest weight λ), then

$$z \cdot \phi(v^+) = \phi(z \cdot v^+) = \phi(\chi_\lambda\, v^+) = \chi_\lambda\, \phi(v^+).$$

Hence, using (2.14), $Y(\lambda)$ admits precisely one central character, namely χ_λ. Finally, notice that there is no guarantee that $\chi_\lambda \neq \chi_\mu$ for $\lambda \neq \mu$. The Harish-Chandra Theorem gives us precise conditions as to when $\chi_\lambda = \chi_\mu$.

We recall from section 1.9 the special element $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha = \sum_{i=1}^{l} w_i \in \mathcal{H}^*$. If $\lambda, \mu \in \mathcal{H}^*$, we say that λ is **linked** to μ, denoted $\lambda \sim \mu$, if there exists $\sigma \in \mathcal{W}$ such that $\sigma(\lambda+\delta) = \mu + \delta$.

The main theorem regarding central characters can now be formulated as follows.

<u>**Theorem 2.15 (Harish-Chandra):**</u> *Let $\lambda, \mu \in \mathcal{H}^*$. Then $\chi_\lambda = \chi_\mu$ iff $\lambda \sim \mu$.*

<u>**Proof:**</u> (see [Hu 1] §23.2–3). ∎

Despite the fact that we will not prove the above theorem, we will be needing a certain map which is used in the proof. First we need some notation. Let $\mathfrak{Z}$ be the center of $\mathfrak{U}(L)$ and let $\mathfrak{U}(\mathcal{H})^{\mathcal{W}}$ denote the algebra consisting of those elements of $\mathfrak{U}(\mathcal{H})$ which are invariant under the action of the Weyl group $\mathcal{W}$. We define the **Harish-Chandra isomorphism** to be the map $\psi\colon \mathfrak{Z} \to \mathfrak{U}(\mathcal{H})^{\mathcal{W}}$ given as follows.

Fix a basis of L consisting of $\{h_i \mid 1 \leq i \leq l\}$ along with $\{x_\alpha, y_\alpha \mid \alpha \in \Phi^+\}$. Construct P-B-W bases for $\mathfrak{U}(L)$ and $\mathfrak{U}(\mathcal{H})$, relative to the ordering on L (and hence on $\mathcal{H}$) putting the y_α first, then the h_i, followed by the x_α. Define a linear map $\xi\colon \mathfrak{U}(L) \to \mathfrak{U}(\mathcal{H})$ by sending any basis element consisting only of h_i's to itself and all other basis elements to 0. We also define a map η by sending $h_i \to h_i - 1$ and extending linearly. This is a Lie algebra homomorphism on $\mathcal{H}$ (since all commutator products are 0) and so can be extended uniquely to a homomorphism $\eta: \mathfrak{U}(\mathcal{H}) \to \mathfrak{U}(\mathcal{H})$ having the property that $\eta: 1 \to 1$. It is clear that η is an isomorphism. Finally, ψ is defined to be $\psi = (\eta \circ \xi) \downarrow_{\mathfrak{Z}}$.

2.5 Formal Characters

As before we let P be the lattice of integral weights, i.e. the lattice of integral linear functionals on $\mathcal{H}$. We wish to introduce formal sums of weights for standard cyclic modules which make it easier to recognize components in a direct sum decomposition of various modules.

We begin by considering finite dimensional modules. To this end, we consider P as an additive abelian group having free generators $\{\omega_1, \ldots, \omega_n\}$. To ensure that our computations make sense, we introduce $\mathbb{Z}[P]$, the group ring of P over $\mathbb{Z}$. By definition, $\mathbb{Z}[P]$ is a free $\mathbb{Z}$-module with basis elements $e(\lambda)$ which are in one-to-one correspondence with the $\lambda \in P$. We denote addition in $\mathbb{Z}[P]$ by $e(\lambda) + e(\mu)$ and define multiplication in $\mathbb{Z}[P]$ by demanding that $e(\lambda)\, e(\mu) = e(\lambda + \mu)$ and extending linearly. In particular, this definition ensures that $\mathbb{Z}[P]$ is a commutative ring with multiplicative identity $e(0)$. Also, there is a natural action of $\mathcal{W}$ on $\mathbb{Z}[P]$, which is given by $\sigma \cdot e(\lambda) = e(\sigma \cdot \lambda)$, (i.e. $\mathcal{W}$ permutes the $e(\lambda)$.

Now let $V = V(\lambda)$ be a finite dimensional irreducible standard cyclic module (i.e. $\lambda \in P^+$). We define the **formal character** of V, which we denote by $\mathrm{ch}_{V(\lambda)}$ or ch_λ, to be the sum $\sum_{\mu \in P} m_\lambda(\mu)\, e(\mu)$ in $\mathbb{Z}[P]$. Since $m_\lambda(\mu) = 0$ for $\mu \notin \Pi(\lambda)$, this is in fact a finite sum. We can immediately generalize this to all finite dimensional modules. If V is any finite dimensional L-module, then Weyl's Theorem implies that we may decompose V as a direct sum $\sum_{i=1}^{t} \oplus V(\lambda_i)$ of finite dimensional irreducible standard cyclic modules. Since this decomposition is unique up to the order of factors, it thus makes sense to define the formal character of V to be $\mathrm{ch}_V = \sum_{i=1}^{t} \mathrm{ch}_{\lambda_i}$. Notice that, in fact, $\mathrm{ch}_V = \sum_{\mu \in P} m_\lambda(\mu)\, e(\mu)$. Note also that since each $\sigma \in \mathcal{W}$ permutes weight spaces in each $V(\lambda_i)$, σ leaves ch_V fixed.

The following result allows us to retrieve the irreducible components of a finite dimensional L-module V, given its formal character.

Proposition 2.16: *Let $f \in \mathbb{Z}[P]$ be invariant under $\mathcal{W}$. Then f can be uniquely expressed as an integral linear combination of ch_λ, where $\lambda \in P^+$.*

Proof: Any such f can be written as $f = \sum_{\mu \in P} c_\mu\, e(\mu)$ where all but finitely many c_μ are zero.

Since f is $\mathcal{W}$-invariant, we can rewrite this as

$$f = \sum_{\lambda \in P^+} c_\lambda \sum_{\substack{\text{distinct } \sigma\lambda \\ \sigma \in \mathcal{W}}} e(\sigma\lambda)$$

Define $M_f = \{\, \mu \in P^+ \mid \text{there exists } \lambda \in P^+ \text{ with } c_\lambda \neq 0 \text{ and } \mu \prec \lambda \,\}$. Then $|M_f| < \infty$.

We prove the existence of such an expression by inducting on $|M_f|$. If $|M_f| = 0$, then clearly $f = 0$. If $|M_f| > 0$, we select $\lambda \in M_f$ with λ maximal with respect to $\prec$ and let $f' = f - c_\lambda\, \mathrm{ch}_\lambda$. Then f' is again invariant under $\mathcal{W}$ and $|M_{f'}| < |M_f|$, since $M_{f'} \subsetneq M_f$. By induction, f' can be written uniquely as an integral linear combination of $\mathrm{ch}_{\lambda'}$. Hence so can $f = f' + c_\lambda\, \mathrm{ch}_\lambda$. ∎

Proposition 2.17: *If V and W are finite dimensional L-modules, then $\mathrm{ch}_{V \otimes W} = (\mathrm{ch}_V)\,(\mathrm{ch}_W)$.*

Proof: From the way in which L acts on $V \otimes W$, clearly the weights of $V \otimes W$ are of the form $\mu + \nu$, where μ is a weight of V and ν a weight of W. Each such weight $\mu + \nu$ occurs in $V \otimes W$ with multiplicity

$$\sum_{\pi+\pi'=\mu+\nu} m_V(\pi)\, m_W(\pi'). \tag{2.18}$$

On the other hand, the coefficient of $e(\mu+\nu)$ in $(\mathrm{ch}_V)\,(\mathrm{ch}_W)$ is also equal to (2.18).
∎

We now wish to extend these concepts to infinite dimensional modules. Unfortunately, we cannot simply extend the above definition of formal characters, since the manipulation of infinite sums would be difficult. To handle the cases of the infinite dimensional modules which are of interest to us, we now view $\mathbb{Z}[P]$ as the set of $\mathbb{Z}$-valued functions on $\mathcal{H}^*$ which have finite support contained in P. We define

$$\mathcal{X} = \{\, f : \mathcal{H}^* \to \mathbb{Z} \mid \mathrm{supp}(f) \subseteq \bigcup_{\substack{\text{finitely} \\ \text{many } \lambda}} K_\lambda \,\}$$

where $\mathrm{supp}(f) = \{\, \mu \in \mathcal{H}^* \mid f(\mu) \neq 0 \,\}$ and

$$K_\lambda = \{\, \lambda - \sum_{\alpha \in \Phi^+} k_\alpha\, \alpha \mid k_\alpha \in \mathbb{Z}^+ \,\}.$$

In particular, we let $e_\mu : \mathcal{H}^* \to \mathbb{Z}$ be defined by setting $e_\mu(\nu) = 1$ if $\nu = \mu$ and $e_\mu(\nu) = 0$ if $\nu \neq \mu$. We also define multiplication in $\mathcal{X}$ to be convolution: for $f, g \in \mathcal{X}$, $f * g(\mu) = \sum_{\pi+\pi'=\mu} f(\pi)\, g(\pi')$. In particular, $e_\mu * e_\nu = e_{\mu+\nu}$. Also, $\mathcal{W}$ acts naturally on elements of $\mathcal{X}$ by $\sigma(e_\mu) = e_{\sigma(\mu)}$. Hence in particular we may consider the e_μ as replacing the $e(\mu)$ used above. In the remaining sections we will do so.

We should also show that $\mathcal{X}$ is actually closed under convolution. But if $f, g \in \mathcal{X}$ with $\mathrm{supp}(f) \subseteq \cup_{i=1}^{n} K_{\lambda_i}$ and $\mathrm{supp}(g) \subseteq \cup_{j=1}^{m} K_{\lambda'_j}$, then clearly $\mathrm{supp}(f * g) \subseteq \cup_{i,j} K_{\lambda_i + \lambda'_j}$. Thus $\langle \mathcal{X}, +, * \rangle$ is an associative, commutative algebra with multiplicative identity e_0.

Now, for an arbitrary L-module V having weights lying in finitely many sets of the form K_λ, we may define the **formal character** of V to be $\mathrm{ch}_V = \sum_{\mu \in \mathcal{H}^*} m_V(\mu) e_\mu$. Note that this definition agrees with the previous one for finite dimensional modules. Also, any standard cyclic module has weights lying in precisely one set of the form K_λ, and so its formal character is defined. For any $\lambda \in \mathcal{H}^*$, we denote by ch_λ the formal character of the irreducible standard cyclic module $V(\lambda)$ and by ch'_λ the formal character of the Verma module $Z(\lambda)$.

It is quite useful to note that if λ and μ are distinct elements of $\mathcal{H}^*$, at most one of $m_{Z(\lambda)}(\mu)$ and $m_{Z(\mu)}(\lambda)$ can be nonzero. In particular, this implies that the various ch'_λ,for $\lambda \in \mathcal{H}^*$, are linearly independent over $\mathbb{Z}$.

In order to simplify manipulations of formal characters, we need to introduce a couple of special functions of $\mathcal{X}$. Let $p(\nu)$ equal the number of sequences $(k_\alpha)_{\alpha \in \Phi^+}$ of nonnegative integers such that $\nu = -\sum_{\alpha \in \Phi^+} k_\alpha \alpha$. Clearly this makes p into a function $\mathcal{H}^* \to \mathbb{Z}$ such that for any $f \in \mathcal{X}$ which is outside the negative integer lattice, $p(f) = 0$. An interesting property of p, which is sometimes called the translation principle, is given in the following lemma.

Proposition 2.19: *Let $\lambda, \mu \in \mathcal{H}^*$. Then*

$$ch_{Z(\lambda)}(\mu) = p(\mu - \lambda) = (p * e_\lambda)(\mu). \tag{2.20}$$

Proof: The first equality follows directly from the definitions of $Z(\lambda)$ and p. For the second equality, notice that

$$\begin{aligned}(p * \varepsilon_\lambda)(\mu) &= \prod_{\pi+\pi'=\mu} p(\pi)\,\varepsilon_\lambda(\pi') \\ &= \prod_{\pi=\mu-\lambda} p(\pi) = p(\mu-\lambda). \quad \blacksquare\end{aligned}$$

2.6 Contravariant Forms on Standard Cyclic L-modules

We now wish to introduce the concept of a contravariant form on a highest weight L-module and prove some results which turn out to be crucial to the remaining work. Throughout this section, let L denote a simple Lie algebra over $\mathbb{C}$ having fixed Cartan subalgebra $\mathcal{H}$ and universal enveloping algebra $\mathfrak{U}$.

Definition 2.21: Let $\sigma: \mathfrak{U} \to \mathfrak{U}$ denote the unique involutive anti-automorphism determined by setting $\sigma(X_\alpha) = X_{-\alpha}$ and $\sigma(H) = H$, where the X_α are fixed elements in the root spaces of L with respect to $\mathcal{H}$ and H is an arbitrary element of $\mathcal{H}$.

Let $\mathfrak{Z}$ be the center of $\mathfrak{U}$ and let $\psi: \mathfrak{Z} \to \mathfrak{U}(\mathcal{H})^W$ be the Harish-Chandra bijection. Let $z \in \mathfrak{Z}$. Then the images under ψ of z and $\sigma(z)$ depend only on the corresponding parts in $\mathfrak{U}(\mathcal{H})$ and these parts are the same for both z and $\sigma(z)$. Thus we have $\psi(z) = \psi(\sigma(z))$. By the injectivity of ψ we conclude that $z = \sigma(z)$ and hence $\sigma\downarrow_{\mathfrak{Z}} =$ id.

Definition 2.22: Let $\Psi: \mathfrak{U}(L) \to \mathfrak{U}(\mathcal{H})$ denote the usual restriction map (i.e select a P-B-W basis of $\mathfrak{U}(L)$ of the form $Y's H's X's$ and then Ψ is the map which kills all basis elements which contain $X's$ or $Y's$ and is the identity on all others.) For any $\lambda \in \mathcal{H}^*$ define the map $f_\lambda: \mathfrak{U} \to \mathbb{C}$ by setting $f_\lambda = \lambda \circ \Psi$.

It is easy to see that $f_\lambda \circ \sigma = f_\lambda$. Indeed, it suffices to consider the action of both maps on basis elements of $\mathfrak{U}$. To this end let

$$u = \prod_{\alpha \succ 0} Y_\alpha^{i_\alpha} \prod_{i=1}^{l} H_i^{k_i} \prod_{\alpha \succ 0} X_\alpha^{j_\alpha}. \tag{2.23}$$

If $\sum_{\alpha \succ 0} i_\alpha = 0 = \sum_{\alpha \succ 0} j_\alpha$ then $\sigma(u) = u$ so $f_\lambda \circ \sigma(u) = f_\lambda(u)$. On the other hand, if $\sum_{\alpha \succ 0} i_\alpha > 0$ or $\sum_{\alpha \succ 0} j_\alpha > 0$ then $\Psi(\sigma(u)) = \Psi(u) = 0$. In either case, we have $f_\lambda \circ \sigma(u) = f_\lambda(u)$, which completes the proof.

Definition 2.24: We define a complex form B_λ on $\mathfrak{U}$ by setting, for all $u, u' \in \mathfrak{U}$, $B_\lambda(u, u') = f_\lambda(\sigma(u)u')$.

Proposition 2.25: *B_λ as defined above is a symmetric, bilinear form on $\mathfrak{U}$.*

Proof: Clearly by its definition f_λ is a linear map. Thus if $a, b \in \mathbb{C}$ and $u_1, u_2, u_1', u_2' \in \mathfrak{U}$ then

$$\begin{aligned} B_\lambda(u_1 + au_2, u_1' + bu_2') &= f_\lambda\left(\sigma(u_1 + au_2)(u_1' + bu_2')\right) \\ &= f_\lambda\big(\sigma(u_1)u_1' + b\sigma(u_1)u_2' \\ &\qquad + a\sigma(u_2)u_1' + ab\sigma(u_2)u_2'\big) \\ &= B_\lambda(u_1, u_1') + bB_\lambda(u_1, u_1') + aB_\lambda(u_2, u_1') \\ &\qquad + abB_\lambda(u_2, u_2') \end{aligned}$$

and so B_λ is bilinear. Furthermore, if $u, u' \in \mathfrak{U}$ then $B_\lambda(u, u') = f_\lambda\left(\sigma(u)u'\right) = f_\lambda \circ \sigma\left(\sigma(u)u'\right) = f_\lambda\left(\sigma(u')\sigma^2(u)\right) = f_\lambda\left(\sigma(u')u\right) = B_\lambda(u', u)$, proving the proposition. ∎

In the following we assume that V is a highest weight L-module with maximal vector v^+ having weight λ. We denote by V' the linear subspace $\sum_{\mu \neq \lambda} V_\mu$ and let Y denote the unique maximal submodule of V.

Lemma 2.26: *For any $u \in \mathfrak{U}$ we have $uv^+ = f_\lambda(u)v^+$ modulo V'. Hence $uv^+ \in Y$ if and only if u is in the kernel of B_λ.*

Proof: Since f_λ is linear, it suffices to consider the action of an element of our P-B-W basis as chosen in Definition (2.22). To this end let u be as in (2.23). If $\sum_{\alpha \succ 0} j_\alpha > 0$ then $uv^+ = 0$. Also, $\Psi(u) = 0$ and hence $f_\lambda(u) = 0$. Thus we can assume that $\sum_{\alpha\succ 0} j_\alpha = 0$. If in addition $\sum_{\alpha \succ 0} i_\alpha = 0$, then $u = \prod_{i=1}^{l} H_i^{k_i}$ and so $f_\lambda(u) = \prod_{i=1}^{l} \lambda(H_i)^{k_i}$ and $uv^+ = (\prod_{i=1}^{l} \lambda(H_i)^{k_i})v^+$. Finally, assume that $\sum_{\alpha\succ 0} i_\alpha > 0$ which implies that $\Psi(u) = 0$ and hence $f_\lambda(u) = 0$. On the other hand, $uv^+ = \prod_{\alpha \succ 0} Y_\alpha^{i_\alpha}(\prod_{i=1}^{l} H_i^{k_i} v^+) = \prod_{i=1}^{l} \lambda(H_i)^{k_i}(\prod_{\alpha\succ 0} Y_\alpha^{i_\alpha} v^+)$. But since $\sum_{\alpha\succ 0} i_\alpha > 0$, we have $\prod_{\alpha\succ 0} Y_\alpha^{i_\alpha} v^+ \in V_{\lambda - \Sigma i_\alpha \alpha} \neq V_\lambda$. Hence $\prod_{\alpha \succ 0} Y_\alpha^{i_\alpha} v^+ \in V'$ and therefore $uv^+ = 0$ mod v'). Since these cases are exhaustive, this proves the first statement.

To prove the second statement, first note that

$$\begin{aligned} u \in \operatorname{Ker} B_\lambda &\iff B_\lambda(u', u) = 0 \quad (\forall u' \in \mathfrak{U}) \\ &\iff f_\lambda\big(\sigma(u')u\big) = 0 \quad (\forall u' \in \mathfrak{U}) \\ &\iff f_\lambda(u'u) = 0 \quad (\forall u' \in \mathfrak{U}) \end{aligned}$$

Hence it suffices to prove that $uv^+ \in Y$ iff $f_\lambda(u'u) = 0$ for all $u' \in \mathfrak{U}$.

Suppose first that $uv^+ \in Y$. Then viewing Y as a $\mathfrak{U}$-module, we have that $\mathfrak{U}(uv^+) \subseteq Y$. Let u' be an arbitrary element of $\mathfrak{U}$. Then $(u'u)v^+ \in Y$ and so, using the first part of the lemma, $(u'u)v^+ = f_\lambda(u'u)v^+$ (mod V'). But $v^+ \notin Y$ and $f_\lambda(u'u) = 0$.

Conversely, suppose that $f_\lambda(u'u) = 0$ for every $u' \in \mathfrak{U}$. Then for all $u' \in \mathfrak{U}$, again by the first part, $u'(uv^+) = 0$ (mod V'). If W denotes the $\mathfrak{U}$-submodule of V generated by uv^+, then $v^+ \notin W$. Hence by the maximality of Y, we must have $W \subseteq Y$ and so $uv^+ \in Y$. ∎

Definition 2.27: Define a form on V (also denoted by B_λ) by setting, for all $v, v' \in V$, $B_\lambda(v, v') = B_\lambda(u, u')$ where $u, u' \in \mathfrak{U}$ are such that $v = uv^+$ and $v' = u'v^+$.

Of course we must check that this form on V is well-defined. To this end suppose that $v = u_1v^+ = u_2v^+$. Then $(u_1 - u_2)v^+ = 0$ and so $\sigma(u')(u_1 - u_2)v^+ = 0$. But then $f_\lambda(\sigma(u')(u_1 - u_2)) = 0$ and this implies that $f_\lambda(\sigma(u')u_1) = f_\lambda(\sigma(u')u_2)$. Thus $B_\lambda(u', u_1) = B_\lambda(u', u_2)$ as required. A similar argument proves that the form is well-defined in the first component.

Definition 2.28: Let V be as given above. Then a **contravariant form** on V is a symmetric, bilinear form B on V satisfying the property that for any $u \in \mathfrak{U}$ and any $v, v' \in V$, $B(uv, v') = B(v, \sigma(u)v')$.

Proposition 2.29: *The form B_λ on V as defined above is a contravariant form on V. Moreover we have that $B_\lambda(v^+, v^+) = 1$ and Y is equal to the kernel of B_λ.*

Proof: By its definition, B_λ is clearly bilinear and symmetric. Also, if $u \in \mathfrak{U}$ and $v_1, v_2 \in V$ with $v_1 = u_1v^+$ and $v_2 = u_2v^+$ then $B_\lambda(uv_1, v_2) = B_\lambda(uu_1, u_2) = f_\lambda(\sigma(uu_1)u_2) = f_\lambda(\sigma(u_1)\sigma(u)u_2) = B_\lambda(u_1, \sigma(u)u_2) = B_\lambda(v_1, \sigma(u)v_2)$. Thus B_λ is a contravariant form on V.

For the second statement, $B_\lambda(v^+, v^+) = B_\lambda(1,1) = f_\lambda(1) = 1$. To see that $Y = \operatorname{Ker} B_\lambda$, first note that the contravariant property of B_λ implies that if $w \in \operatorname{Ker} B_\lambda$ then $B_\lambda(uw, v) = B_\lambda(w, \sigma(u)v) = 0$ for any $u \in \mathfrak{U}$ and any $v \in V$. Hence $\operatorname{Ker} B_\lambda$ is a submodule of V. Moreover, since $B_\lambda(v^+, v^+) = 1$, $v^+ \notin \operatorname{Ker} B_\lambda$. By maximality of Y, $\operatorname{Ker} B_\lambda \subseteq Y$.

Conversely, let $y \in Y$. Then in particular $y \in V$ and so $y = uv^+$ for some $u \in \mathfrak{U}$. Since $uv^+ \in Y$, by the previous lemma, we have $u \in \operatorname{Ker} B_\lambda$ (note that here B_λ is considered as a form on $\mathfrak{U}$). If $v \in V$ then $v = u'v^+$ for some $u' \in \mathfrak{U}$ and hence

$B_\lambda(v, y) = B_\lambda(u', u) = 0$. Thus for all $v \in V$, $B_\lambda(v, y) = 0$ and hence $y \in \operatorname{Ker} B_\lambda$. Therefore $Y \subseteq \operatorname{Ker} B_\lambda$ and we have $Y = \operatorname{Ker} B_\lambda$. ∎

The construction of contravariant forms can be extended to tensor products of modules in a natural way.

<u>**Proposition 2.30:**</u> *Let M_1 and M_2 be L-modules having contravariant forms B_1 and B_2 respectively. Then the 'product' of B_1 and B_2 is a contravariant form on $M_1 \otimes M_2$. In addition, if B_1 and B_2 are nondegenerate then so is the product.*

<u>**Proof:**</u> Let $\{v_i\}$ and $\{w_j\}$ be bases of M_1 and M_2, respectively, so that $\{v_i \otimes w_j\}$ is a basis of $M_1 \otimes M_2$. For each pair of basis vectors $v_i \otimes w_j, v_k \otimes w_l$ define

$$B(v_i \otimes w_j, v_k \otimes w_l) = B_1(v_i, v_k) B_2(w_j, w_l)$$

and extend the construction linearly. By construction then, B is bilinear. To see that B is symmetric, it suffices to prove this for basis elements, and this follows from the definition of B and the fact that B_1 and B_2 are symmetric.

To prove the contravariant property it again suffices, since σ is linear, to prove the property holds on basis elements. With this in mind, let $v_i \otimes w_j, v_k \otimes w_l$ be basis elements of $M_1 \otimes M_2$ and let $u \in \mathfrak{U}$. Then

$$\begin{aligned}
B\big(u(v_i \otimes w_j), v_k \otimes w_l\big) &= B\big(u(v_i) \otimes w_j + v_i \otimes u(w_j), v_k \otimes w_l\big) \\
&= B_1(uv_i, v_k) B_2(w_j, w_l) + B_1(v_i, v_k) B_2(uw_j, w_l) \\
&= B_1\big(v_i, \sigma(u) v_k\big) B_2(w_j, w_l) \\
&\qquad + B_1(v_i, v_k) B_2\big(w_j, \sigma(u) w_l\big) \\
&= B\big(v_i \otimes w_j, (\sigma(u) v_k) \otimes u(w_j), v_k \otimes (\sigma(u) w_l)\big) \\
&= B\big(v_i \otimes w_j, \sigma(u)(v_k \otimes w_l)\big)
\end{aligned}$$

Hence B is a contravariant form on $M_1 \otimes M_2$.

We now show that the second statement holds. Assume B_1 and B_2 are nondegenerate and let $S = \operatorname{Ker} B$. We prove that $S = (0)$.

First note that clearly no simple tensor of the form $v \otimes w$ $(v \in M_1, w \in M_2)$ can lie in S, for if so then the nondegeneracy of B_1 and B_2 allow us to find $v' \in M_1$ and $w' \in M_2$ such that $B_1(v, v') \neq 0$ and $B_2(w, w') \neq 0$. But this implies that $B(v \otimes w, v' \otimes w') \neq 0$, contrary to the definition of S.

Hence we can assume that some vector of the form $y = \sum_{k=1}^{n} a_k\, v_k \otimes w_k$ lies in S, where $v_k \in M_1, w_k \in M_2$ and $n > 1$. Choose such a y with n minimal.

Suppose that there are some v_i, v_j $(1 \leq i \neq j \leq n)$ in this expression which are linearly dependent, i.e. for some $c \neq 0$, $v_i = c v_j$. Then

$$y = v_j \otimes (c\, a_i\, w_i + a_j\, w_j) + \sum_{k \neq i,j} a_k v_k \otimes w_k .$$

This last sum has fewer than n simple tensors, contradicting the minimality of n.

So we may assume that no two of the v_k are linearly dependent.

Consider the linearly independent pair of vectors v_1 and v_2. We wish to first show that the linear functionals $B_1(v_1, \cdot)$ and $B_1(v_2, \cdot)$ are linearly independent. Suppose that this is not the case. Then there exist coefficients b_1 and b_2 such that

$$b_1\, B_1(v_1, \cdot) + b_2\, B_1(v_2, \cdot) = 0$$

and hence $B_1(b_1 v_1 + b_2 v_2, \cdot) = 0$. By the linear independence of v_1 and v_2, however, $b_1 v_1 + b_2 v_2 \neq 0$ and hence $b_1 v_1 + b_2 v_2 \in \operatorname{Ker} B_1$, contrary to the nondegeneracy of B_1. Thus $B_1(v_1, \cdot)$ and $B_1(v_2, \cdot)$ are linearly independent.

We now wish to show that there exists a vector $u \in M_1$ such that $B_1(v_2, u) = 1$ and $B_1(v_1, u) = 0$. Suppose toward a contradiction that no such u exists. Let

$u \in M_1$ be such that $B_1(v_2, u) = 1$ (the nondegeneracy of B_1 implies that such a vector exists). Then $B_1(v_1, u) = c \neq 0$, and $B_1(cv_2, u) = B_1(v_1, u)$. Also, since $S_1 = \operatorname{Ker} B_1(v_1, \cdot)$ and $S_2 = \operatorname{Ker} B_1(v_2, \cdot)$ are subspaces of codimension 1 in M_1, we must have $S_1 = S_2$ and so for any $v \in S_1$, $B_1(cv_2, v) = 0 = B_1(v_1, v)$. Finally, since $M_1 = S_1 + \mathbb{C}u$, where the sum is direct, we have that

$$0 = B_1(cv_2, \cdot) - B_1(v_1, \cdot) = c\,B_1(v_2, \cdot) - B_1(v_1, \cdot),$$

contrary to the linear independence of $B_1(v_1, \cdot)$ and $B_1(v_2, \cdot)$. Thus there exists $u \in M_1$ such that $B_1(v_2, u) = 1$ and $B_1(v_1, u) = 0$.

Finally, since B_2 is nondegenerate, we may choose $w \in M_2$ such that $B_2(w_2, w) \neq 0$. Then

$$\begin{aligned} B(y, u \otimes w) &= \sum_{k=1}^{n} a_k B_1(v_k, u)\, B_2(w_k, w) \\ &= \sum_{k=2}^{n} a_k B_1(v_k, u)\, B_2(w_k, w) \\ &= B(\sum_{k=2}^{n} a_k v_k \otimes w_k, u \otimes w), \end{aligned}$$

which contradicts the minimality of n.

Thus $S = (0)$ and B is nondegenerate, as required. ∎

It is interesting to find that the weight spaces of V are mutually orthogonal with respect to any contravariant form on V and that up to normalization there is a unique contravariant form on V.

__Proposition 2.31:__ *If B is any contravariant form on V then the weight spaces of V are mutually orthogonal with respect to V. Furthermore, if $B(v^+, v^+) = 1$ then $B = B_\lambda$.*

<u>**Proof:**</u> Let B be a contravariant form on V and let μ and ν be distinct weights of V. Then there exists $h \in \mathfrak{U}(\mathcal{H})$ such that $\mu(h) \neq \nu(h)$. If $v \in V_\mu$ and $v' \in V_\nu$, then

$$\begin{aligned} \mu(h)B(v,v') &= B\big(\mu(h)v,v'\big) \\ &= B(h \cdot v, v') \\ &= B\big(v, \sigma(h) \cdot v'\big) \\ &= B(v, h \cdot v') \\ &= B\big(v, \nu(h)v'\big) \\ &= \nu(h)B(v,v') \end{aligned}$$

and hence $\big(\mu(h) - \nu(h)\big)B(v,v') = 0$. Since $\mu(h) \neq \nu(h)$, $B(v,v') = 0$. Thus for any distinct weights μ,ν of V, $B(V_\mu, V_\nu) = 0$.

Now suppose further that $B(v^+,v^+) = 1$. To prove the second statement, it suffices to show that for a given weight μ of V, B and B_λ agree on V_μ. So fix μ and let $v, v' \in V_\mu$. Then we can write $v = uv^+$ and $v' = u'v^+$ where $u = \sum a_k Y_{\beta_m}^{i_{mk}} \ldots Y_{\beta_1}^{i_{1k}}$ and $u' = \sum b_l Y_{\beta_m}^{j_{ml}} \ldots Y_{\beta_1}^{j_{1l}}$. Then $\sigma(u)u' = \sum a_k b_l X_{\beta_1}^{i_{1k}} \ldots X_{\beta_m}^{i_{mk}} Y_{\beta_m}^{j_{ml}} \ldots Y_{\beta_1}^{j_{1l}}$. Now $u'v^+ \in V_\mu$ and so $\sigma(u)u'v^+ \in V_\lambda$. Therefore $\sigma(u)u'v^+ = cv^+$ for some $c \in C$. By Lemma 2.26, we have $c = f_\lambda\big(\sigma(u)u'\big) = B_\lambda(u,u')$. Hence

$$\begin{aligned} B(v,v') &= B(uv^+, u'v^+) \\ &= B\big(v^+, \sigma(u)u'v^+\big) \\ &= B\big(v^+, B_\lambda(u,u')v^+\big) \\ &= B_\lambda(u,u')B(v^+,v^+) \\ &= B_\lambda(v,v'). \end{aligned}$$

Hence $B = B_\lambda$. ∎

In the following, let $Z(\lambda)$ denote the Verma module with highest weight λ and let $V(\lambda)$ denote the irreducible L-module having highest weight λ.

Theorem 2.32: *Let F be a finite dimensional L-module with basis $\{f_1, \ldots, f_n\}$ consisting of vectors having weights $\mu_1, \ldots, \mu_n$ ordered so that $\mu_i \prec \mu_j$ implies that $i \leq j$. If $\lambda \in \mathcal{H}^*$, then $M = Z(\lambda) \otimes F$ has a filtration $M = M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} = (0)$ such that $M_i/M_{i+1} \cong Z(\lambda + \mu_i)$.*

Proof: Let v^+ be a maximal vector in $Z(\lambda)$ and define $a_i = v^+ \otimes f_i$. Let M_i be the submodule of M generated by $\{a_i, a_{i+1}, \ldots, a_n\}$ and let b_i be the image of a_i in M_i/M_{i+1}. The proof is in stages.

We first show $M_1 = M$. For each i we have $a_i = v^+ \otimes f_i \in M_1$ and hence for every $f \in F$, $v^+ \otimes f \in M_1$. We show that $uv^+ \otimes f \in M_1$ for all monomials $u \in \mathfrak{U}$ by inducting on the degree of u. If u is a degree 1 monomial in $\mathfrak{U}$ then

$$u \cdot (v^+ \otimes f) = (u \cdot v^+) \otimes f + v^+ \otimes (u \cdot f).$$

Since $u \cdot (v^+ \otimes f)$ and $v^+ \otimes (u \cdot f)$ are in M_1, so also is $(u \cdot v^+) \otimes f$. Assume inductively that for any monomial $u \in \mathfrak{U}$ of degree $< n$, $(u \cdot v^+) \otimes f \in M_1$. If $u \in \mathfrak{U}$ is a monomial of degree n, then

$$\begin{aligned} u \cdot (v^+ \otimes f) = (u \cdot v^+) \otimes f \\ &+ [\text{combination of terms of the form } (u' \cdot v^+) \otimes f' \\ &\qquad \text{where } \deg u' < n] \end{aligned}$$

and hence, using the inductive hypothesis, we have that $(u \cdot v^+) \otimes f \in M_1$. Thus by induction, $(u \cdot v^+) \otimes f \in M_1$ for all monomials $u \in \mathfrak{U}$. Since the monomials span $\mathfrak{U}$ and since v^+ generates $Z(\lambda)$, we have $w \otimes f \in M_1$ for all $w \in Z(\lambda)$ and all $f \in F$. Hence $M_1 = M$.

Secondly, if $h \in \mathcal{H}$, then $h \cdot a_i = (h \cdot v^+) \otimes f_i + v^+ \otimes (h \cdot f_i) = (\lambda(h)v^+) \otimes f_i + v^+ \otimes (\mu_i(h) f_i) = (\lambda + \mu_i)(h)(v^+ \otimes f_i)$. Hence each a_i has weight $\lambda + \mu_i$ and so each b_i does also. Moreover, each b_i generates M_i/M_{i+1} as a $\mathfrak{U}$-module.

Next we show that each b_i is maximal. To this end let $\alpha \in \Phi^+$. Then $X_\alpha \cdot a_i = v^+ \otimes (X_\alpha \cdot f_i)$. Now $X_\alpha \cdot f_i \in F_{\mu_i + \alpha}$ which implies that either $X_\alpha \cdot f_i = 0$ or $X_\alpha \cdot f_i = \sum_{j>i} c_j f_j$, where $c_j \in \mathbb{C}$. Hence either $X_\alpha \cdot a_i = 0$ or $X_\alpha \cdot a_i = \sum_{j>i} c_j (v^+ \otimes f_j) = \sum_{j>i} c_j a_j$. In any case, $X_\alpha \cdot a_i \in M_{i+1}$ and so $X_\alpha \cdot (a_i + M_{i+1}) = 0 + M_{i+1}$. In other words, $X_\alpha \cdot b_i = 0$. But α was an arbitrary positive root and so $X_\alpha \cdot b_i = 0$ for all $\alpha \in \Phi^+$. Hence either $b_i = 0$ (in which case the quotient $M_i/M_{i+1} = (0)$) or b_i is a maximal vector in M_i/M_{i+1}.

To see that M_i/M_{i+1} is nonzero, we actually prove the final statement that each quotient is isomorphic to a Verma module. Indeed, using formal characters, let

$$\mathrm{ch}_F = \sum_{\mu_i \text{ distinct}} m(\mu_i)\, e_{\mu_i} = \sum_{i=1}^{n} e_{\mu_i}$$

where in the last sum the μ_i need not be distinct. Then

$$\begin{aligned}
\mathrm{ch}_{Z(\lambda)\otimes F} &= \mathrm{ch}_{Z(\lambda)} * \mathrm{ch}_F \\
&= \mathrm{ch}_{Z(\lambda)} * (\sum_{i=1}^{n} e_{\mu_i}) \\
&= \sum_{i=1}^{n} \mathrm{ch}_{Z(\lambda)} * e_{\mu_i} \\
&= \sum_{i=1}^{n} p * e_\lambda * e_{\mu_i} \\
&= \sum_{i=1}^{n} p * e_{\lambda+\mu_i} \\
&= \sum_{i=1}^{n} \mathrm{ch}_{Z(\lambda+\mu_i)}
\end{aligned}$$

and hence we have that $M_i/M_{i+1} \cong Z(\lambda + \mu_i)$. ∎

We now concentrate on tensoring $V(\lambda)$ with a finite dimensional module.

Keeping the previous notation, we note that $M = V(\lambda) \otimes V(\lambda_0)$ is a quotient of $Z(\lambda) \otimes V(\lambda_0)$ and therefore has a filtration with quotients which are highest weight modules having weights of the form $\lambda + \mu$ where μ is a weight of $V(\lambda_0)$. Hence the possible distinct central characters which can occur in M must be of the form $\chi_{\lambda+\mu_i}$ where the μ_i run over a subset of the weights of $V(\lambda_0)$. Form the canonical decomposition $M = M^{(1)} \oplus \cdots \oplus M^{(t)}$ into generalized eigenspaces corresponding to distinct central characters of M. Now if p_i is the projection of M onto $M^{(i)}$ then we obtain a filtration of $M^{(i)}$ with quotients which are highest weight modules belonging to linked weights $\lambda + \mu = \lambda + \mu_i$. If μ is maximal among the weights of $V(\lambda_0)$ which occur this way it is clear that the corresponding vectors in $M^{(i)}$ are actually maximal vectors. In any case $M^{(i)}$ is generated by the projections of those spaces $v^+ \otimes V(\lambda_0)_\mu$ for which $\lambda + \mu$ is linked to $\lambda + \mu_i$.

__Theorem 2.33:__ *(a) The contravariant forms on $V(\lambda)$ and $V(\lambda_0)$ are nondegenerate (and hence by Prop. 2.30 yield a nondegenerate contravariant form on the tensor product). Moreover, with respect to this form, the submodules $M^{(i)}$ are orthogonal, and hence the form is nondegenerate on each.*

(b) Suppose that $\mu = \mu_i$ is a weight of $V(\lambda_0)$, such that for all weights $\nu \neq \mu$ of $V(\lambda_0)$, $\lambda + \nu$ is not linked to $\lambda + \mu$. Then $M^{(i)}$ is the direct sum of n copies of $V(\lambda + \mu)$, where $n = \dim M^{(i)}_{\lambda+\mu}$.

__Proof:__ (a) Let B_1, B_2 be the given contravariant forms on $V(\lambda)$ and $V(\lambda_0)$, respectively. Since $V(\lambda)$ and $V(\lambda_0)$ are irreducible, by Prop. 2.29 we have that $\operatorname{Ker} B_1 = \operatorname{Ker} B_2 = (0)$. Therefore B_1 and B_2 are nondegenerate and Prop. 2.30

applies to give that the product form B on $V(\lambda)\otimes V(\lambda_0)$ is nondegenerate.

We now concentrate on proving that the $M^{(i)}$ are orthogonal with respect to B. From this it will follow immediately that B is nondegenerate on each $M^{(i)}$. So consider two generalized eigenspaces $M^{(i)}\neq M^{(j)}$ with corresponding central characters $\chi_{\lambda+\mu_i}\neq\chi_{\lambda+\mu_j}$. Then there is a $z\in\mathfrak{Z}$ such that $\chi_{\lambda+\mu_i}(z)=a_i\neq a_j=\chi_{\lambda+\mu_j}(z)$.

Suppose now that μ' is a weight for which $M^{(i)}_{\mu'}\neq(0)$ and $M^{(j)}_{\mu'}\neq(0)$ and let $v_i\in M^{(i)}_{\mu'}, v_j\in M^{(j)}_{\mu'}$. In addition, let k_i and k_j be minimal such that $(z-a_i)^{k_i}v_i=0$ and $(z-a_j)^{k_j}v_j=0$. We show $B(v_i,v_j)=0$ by induction on $\ell=k_i+k_j$.

If $\ell=2$, then either (i) $k_i=2,k_j=0$, (ii) $k_i=0,k_j=2$, or (iii) $k_i=k_j=1$. In case (i) holds, we have that $v_j=0$ and hence $B(v_i,v_j)=0$. Similarly for case (ii). In case (iii) holds, both v_i and v_j are eigenvectors for z, and so

$$\begin{aligned}a_iB(v_i,v_j)&=B(z\cdot v_i,v_j)\\&=B(v_i,\sigma(z)\cdot v_j)\\&=B(v_i,z\cdot v_j)\\&=a_jB(v_i,v_j)\end{aligned}$$

which forces $B(v_i,v_j)=0$.

Assume inductively that the result holds for any vectors v_i',v_j' satisfying $2\leq\ell<m$ and suppose that $\ell=m$. First, $(z-a_i)v_i$ and v_j are vectors in $M^{(i)}_{\mu'}$ and $M^{(j)}_{\mu'}$, respectively, such that $(z-a_i)^{k_i-1}\big((z-a_i)v_i\big)=0=(z-a_j)^{k_j}v_j$ and with exponents summing to $\ell-1$. By the minimality of k_i and k_j, the inductive hypothesis applies to give $B(z\cdot v_i,v_j)=a_iB(v_i,v_j)$. A similar argument gives $B(v_i,z\cdot v_j)=a_jB(v_i,v_j)$. Arguing as in the case $\ell=2$, this implies that $(a_i-a_j)B(v_i,v_j)=0$ and hence that $B(v_i,v_j)=0$.

Thus by induction $B(M^{(i)}_{\mu'}, M^{(j)}_{\mu'}) = 0$. By Proposition. 2.31, $B(M^{(i)}_{\mu'}, M^{(j)}_{\mu''}) = 0$ for any distinct weights μ', μ''. Combining these we get $B(M^{(i)}, M^{(j)}) = 0$ for $i \neq j$, as required.

(b) Let $c(\mu) = \dim V(\lambda)_\mu$ and let $\{v_1, \ldots, v_{c(\mu)}\}$ be a basis of $V(\lambda)_\mu$. Let w_k be the projection of $v^+ \otimes v_k$ in $M^{(i)}$. By the remarks preceding the theorem, since $\mu \neq \nu$ in $V(\lambda_0)$ implies that $\lambda + \mu \not\prec \lambda + \nu$, μ is trivially maximal among the weights of $V(\lambda_0)$ and so the corresponding w_k are maximal vectors in $M^{(i)}$.

Now, using part (a), the contravariant form B on $V(\lambda) \otimes V(\lambda_0)$ is nondegenerate on $M^{(i)}$ and hence B is nondegenerate on the weight space $M^{(i)}_{\lambda+\mu}$. Let $n = \dim M^{(i)}_{\lambda+\mu}$. Clearly $n \leq c(\mu)$. Reorder the w_k's so that $\{w_1, w_2, \ldots, w_n\}$ form a basis of $M^{(i)}_{\lambda+\mu}$ consisting of maximal vectors. Let $c_k = B(w_k, w_k) \neq 0$ and set $w'_k = \frac{1}{\sqrt{c_k}} w_k$ so $B(w'_k, w'_k) = 1$. Then $\{w'_1, w'_2, \ldots w'_n\}$ is an orthogonal basis of maximal vectors. Clearly, the set $\{\prod_{\alpha \succ 0} Y_\alpha^{i_\alpha} w'_k | i_\alpha \in Z^+;\ 1 \leq k \leq n\}$ spans $M^{(i)}$ and so the w'_k's generate $M^{(i)}$ as a $\mathfrak{U}(\mathfrak{N}^-)$-module.

Furthermore, each subspace $\mathfrak{U}(\mathfrak{N}^-)\, w'_k$ is a highest weight module with maximal vector w'_k of weight $\lambda + \mu$. Also, each pair $\mathfrak{U}(\mathfrak{N}^-)\, w'_i$ and $\mathfrak{U}(\mathfrak{N}^-)\, w'_j$ $(i \neq j)$ is orthogonal with respect to B and hence B is nondegenerate on each $\mathfrak{U}(\mathfrak{N}^-)\, w'_k$. In particular then, this says that for $i \neq j$, $\mathfrak{U}(\mathfrak{N}^-)\, w'_i \cap \mathfrak{U}(\mathfrak{N}^-)\, w'_j = (0)$.

Finally, since w'_k is a maximal vector for $\mathfrak{U}(\mathfrak{N}^-)\, w'_k$, and since $B(w'_k, w'_k) = 1$, Prop. 2.30 says the contravariant form B resticted to $\mathfrak{U}(\mathfrak{N}^-)\, w'_k$ is unique. In addition, since $\operatorname{Ker} B = (0)$, Prop. 2.28 applies to give that $\mathfrak{U}(\mathfrak{N}^-)\, w'_k$ is irreducible of highest weight $\lambda + \mu$. Thus $\mathfrak{U}(\mathfrak{N}^-)\, w'_k \cong V(\lambda + \mu)$. Also, the w'_k's generate $M^{(i)}$ and the $\mathfrak{U}(\mathfrak{N}^-)\, w'_k$'s are pairwise disjoint and so we conclude that

$$M^{(i)} \cong \sum_{j=1}^{n} \oplus V(\lambda + \mu)$$

(where as above $n = \dim M^{(i)}_{\lambda+\mu}$). ∎

Chapter 3

Completely Pointed C_n-Modules

3.1 Actions of C_n

Let us now restrict our attention to the simple Lie algebras of type C_n. Throughout this section, let $\mathcal{H}$ be a fixed Cartan subalgebra of C_n. As in section 1.8, we let $(\cdot, \cdot)$ denote the usual inner product on $\mathbb{R}^n$.

The root system of C_n can be identified with the set of vectors of the form

$$\{\pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i < j \leq n\} \cup \{\pm 2\epsilon_i \mid i = 1, \ldots, n\}$$

in $\mathbb{R}^n$ where ϵ_i denotes the ith standard basis vector of $\Re^n$. Note that the ϵ_i's form an orthonormal basis for $\mathbb{R}^n$ with respect to $(\cdot, \cdot)$. We choose a simple base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ for the root system of C_n as follows.

$$\alpha_i = \epsilon_{n-i} - \epsilon_{n-i+1} \qquad (1 \leq i \leq n-1)$$

$$\alpha_n = -2\epsilon_1$$

We can also calculate the set $\Omega = \{\omega_1, \ldots, \omega_n\}$ of fundamental dominant weights corresponding to the simple base Δ. Recall that a base of simple roots and its corresponding fundamental dominant weights satisfy

$$\langle \omega_i, \alpha_j \rangle = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

where δ_{ij} is the Kronecker delta operator. Using this, we can calculate the fundamental dominant weights in terms of the ϵ_i's. We get:

$$\omega_i = -\Big(\sum_{j=n-i+1}^{n} \epsilon_j \Big) \qquad (1 \le i \le n) \tag{3.1}$$

Recall that since the Killing form is nondegenerate on C_n and on $\mathcal{H}$, therefore to each $\theta \in \mathcal{H}^*$ there is associated a unique element $t_\theta \in \mathcal{H}$ satisfying

$$K(h, t_\theta) = \theta(h) \qquad \text{for all } h \in \mathcal{H}.$$

Then corresponding to the basis Δ of $\mathcal{H}^*$ is the basis $\{H_i\}$ of $\mathcal{H}$ where

$$H_i = \frac{2t_{\alpha_i}}{(\alpha_i, \alpha_i)} \qquad i = 1, \ldots, n$$

In Section 1.4 we defined C_n as a subalgebra of the Lie algebra $sl(2n, \mathbb{C})$ of $2n \times 2n$ traceless complex matrices. Recall that $\{ E_{i,j} \mid 1 \le i, j \le 2n \}$ denotes the basis of $gl(2n, \mathbb{C})$ consisting of standard matrix units. For each i we choose a vector $X_{\alpha_i} \in (C_n)_{\alpha_i}$ and $Y_{\alpha_i} = X_{-\alpha_i} \in (C_n)_{-\alpha_i}$ as follows.

$$X_{\alpha_i} = E_{i,i+1} - E_{n+i+1,n+i}, \qquad i = 1, \ldots, n-1$$

$$X_{\alpha_n} = E_{n,2n}$$

$$Y_{\alpha_i} = X^t_{\alpha_i} \qquad i = 1, \ldots, n$$

We also set

$$H_i = H_{\alpha_i} = E_{i,i} - E_{n+i,n+i} - E_{i+1,i+1} + E_{n+i+1,n+i+1}$$

$$i = 1, \ldots, n-1$$

$$H_n = H_{\alpha_n} = E_{n,n} - E_{2n,2n}.$$

It can be shown that these elements generate C_n as a Lie algebra. Also, for each i the subalgebra of C_n generated by

$$\{X_{\alpha_i}, H_i, Y_{\alpha_i}\}$$

is isomorphic to $sl(2,\mathbb{C})$ via the isomorphism

$$X_{\alpha_i} \longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$H_i \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$Y_{\alpha_i} \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

For ease of notation, when we consider C_2, we set $\alpha = \alpha_1$ and $\beta = \alpha_2$.

Recall that by Proposition 2.4, $\mathfrak{U}_0 = \mathfrak{U}_0(C_n)$ is a finitely generated subalgebra of $\mathfrak{U}(C_n)$. We call $\mathfrak{U}_0(C_n)$ the **cycle subalgebra** of $\mathfrak{U}(C_n)$. In the case of C_2, it turns out that $\mathfrak{U}_0$ is generated by the Cartan elements H_α and H_β, along with the following elements.

$$\begin{aligned}
c_1 &= Y_\alpha\, X_\alpha & c_7 &= Y_{2\alpha+\beta}\, X_\alpha\, X_{\alpha+\beta} \\
c_2 &= Y_\beta\, X_\beta & c_8 &= Y_{\alpha+\beta}\, Y_\alpha\, X_{2\alpha+\beta} \\
c_3 &= Y_{\alpha+\beta}\, X_{\alpha+\beta} & c_9 &= Y_{2\alpha+\beta}\, X_\beta\, X_\alpha\, X_\alpha \\
c_4 &= Y_{2\alpha+\beta}\, X_{2\alpha+\beta} & c_{10} &= Y_\alpha\, Y_\alpha\, Y_\beta\, X_{2\alpha+\beta} \\
c_5 &= Y_{\alpha+\beta}\, X_\beta\, X_\alpha & c_{11} &= Y_{2\alpha+\beta}\, Y_\beta\, X_{\alpha+\beta}\, X_{\alpha+\beta} \\
c_6 &= Y_\alpha\, Y_\beta\, X_{\alpha+\beta} & c_{12} &= Y_{\alpha+\beta}\, Y_{\alpha+\beta}\, X_\beta\, X_{2\alpha+\beta}
\end{aligned}$$

We call $c_1, \ldots, c_{12}$ the **basic cycles** of C_2.

We now define a realization of C_n consisting of operators on $\mathbb{C}[x_1, \ldots, x_n]$, where the x_i's are commuting variables. Denote by W_n the associative subalgebra of $\mathrm{End}_{\mathbb{C}}\mathbb{C}[x_1,\ldots,x_n]$ generated by $\{x_i, \partial_i \mid = 1,\ldots,n\}$ where x_i is viewed as left multiplication by x_i and ∂_i is viewed as partial differentiation with respect to x_i. W_n is called the **Weyl algebra** of rank n. We shall give a Lie embedding of C_n into the subalgebra of W_n consisting of degree 2 elements.

Using the basis of C_n given above, we define a map $\psi: C_n \to W_n$ given explicitly by

$$\psi(X_{\epsilon_i-\epsilon_{i+1}}) = x_i\partial_{i+1} \qquad 1 = 1,\ldots,n-1$$

$$\psi(X_{-(\epsilon_i-\epsilon_{i+1})}) = x_{i+1}\partial_i \qquad 1 = 1,\ldots,n-1$$

$$\psi(X_{-2\epsilon_1}) = -\frac{1}{2}\partial_1^2$$

$$\psi(X_{2\epsilon_1}) = \frac{1}{2}x_1^2$$

$$\psi(H_i) = x_{n-i}\partial_{n-i} - \partial_{n-i+1}x_{n-i+1} \qquad i = 1,\ldots,n-1$$

$$\psi(H_n) = -(x_1\partial_1 + \frac{1}{2})$$

This map determines a monomorphism of C_n into W_n.

For the sake of simplicity, in the remaining sections we drop the reference to the map ψ, but the context will make it clear which action is being used.

For later reference we list explicitly the actions of C_2.

Table 3.2: Actions of C_2

		ψ
X_α Y_α	$E_{12} - E_{43}$ $E_{21} - E_{34}$	$x_1\partial_2$ $x_2\partial_1$
X_β Y_β	E_{24} E_{42}	$-\frac{1}{2}\partial_1^2$ $\frac{1}{2}x_1^2$
$X_{\alpha+\beta}$ $Y_{\alpha+\beta}$	$E_{14} + E_{23}$ $E_{41} + E_{32}$	$\partial_1\partial_2$ $-x_1x_2$
$X_{2\alpha+\beta}$ $Y_{2\alpha+\beta}$	$2E_{13}$ $2E_{31}$	$-\partial_2^2$ x_2^2
H_α H_β	$E_{11} - E_{22} - E_{33} + E_{44}$ $E_{22} - E_{44}$	$x_1\partial_1 - x_2\partial_2$ $-(x_1\partial_1 + \frac{1}{2})$

Finally, we mention one other useful fact. As in Section 1.9, we let $\sigma_i = \sigma_{\alpha_i} \in \mathcal{W}$ be the reflection in the plane perpendicular to α_i. For $i = 1, 2, \ldots, n-1$,

$$\sigma_i(\epsilon_j) = \epsilon_j - (\epsilon_j, \epsilon_{n-i} - \epsilon_{n-i+1})(\epsilon_{n-i} - \epsilon_{n-i+1}).$$

If $j = n - i$, then $\sigma_i(\epsilon_j) = \epsilon_{j+1}$ and if $j = n - i + 1$, $\sigma_i(\epsilon_j) = \epsilon_{j-1}$. Otherwise, $\sigma_i(\epsilon_j) = \epsilon_j$. Also,

$$\sigma_n(\epsilon_j) = \epsilon_j + (\epsilon_j, -2\epsilon_1)\,\epsilon_1.$$

Then $\sigma_n(\epsilon_1) = -\epsilon_1$ and if $j \neq 1$, $\sigma_n(\epsilon_j) = \epsilon_j$. In particular, then, the reflections $\sigma_1, \ldots, \sigma_n$ act on the basis $\{\epsilon_i\}$ by permuting the subscripts and by changing signs. Since the σ_i generate $\mathcal{W}$, we have that every element of $\mathcal{W}$ acts on the ϵ_i's by sign changes and by permuting the subscripts.

3.2 Construction of Completely Pointed C_n-Modules

In this section we construct two infinite-dimensional highest weight C_n-modules in which all weight spaces are one dimensional. We then prove that up to isomorphism these are the only such modules. Keep the notation of Section 3.1. Recall that $\mathbb{Z}^+$ denote the set of nonnegative integers.

Consider the action of C_n on $\mathbb{C}[x_1,\ldots,x_n]$ induced by ψ (i.e. $z \cdot f = \psi(z)f$ for all $z \in C_n$ and $f \in \mathbb{C}[x_1,\ldots,x_n]$). Let $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ be an arbitrary monomial in $\mathbb{C}[x_1\ldots,x_n]$. Notice that by definition of ψ we have $H_{\alpha_i}(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = (a_{n-i} - a_{n-i+1})x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ $(1 \leq i \leq n-1)$ and $H_{\alpha_n}(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = -(a_1 + \frac{1}{2})x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ so that for each choice of monomial of the form $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ we have a unique $\nu \in \mathcal{H}^*$ such that $H(x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}) = \nu(H)x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ for all $H \in \mathcal{H}$ and moreover monomials with distinct exponents clearly give rise to distinct elements of $\mathcal{H}^*$.

Now consider the $\mathfrak{U}$-module M ($\subseteq \mathbb{C}[x_1,\ldots,x_n]$) generated by the monomial 1. From the action of C_n on $\mathbb{C}[x_1,\ldots,x_n]$, it is clear that the set $\{x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \mid a_i \in \mathbb{Z}^+; \sum a_i \text{ even}\}$ forms a basis for M. Also, each monomial in this basis is a weight vector (having weight as determined above) and since these weights are distinct, we have that the weight spaces of M are one-dimensional and M is a direct sum of its weight spaces. In addition, since $X_\alpha(1) = 0$ for all $\alpha \in \Phi^+$, 1 is a maximal vector of M, having weight $\nu = -\frac{1}{2}\omega_n$. Thus M is a standard cyclic module of highest weight $-\frac{1}{2}\omega_n$. Note that since $Y_{\alpha_1} \cdot 1 = 0$, M is not a Verma module. Finally, we note that M is simple. To see this, assume that M_0 is a nonzero submodule of M. Then M_0 contains a nonzero vector of the form $v = \sum_{i=1}^n c_i x_1^{a_{1i}}x_2^{a_{2i}}\cdots x_n^{a_{ni}}$ where the c_i's are constants and each monomial is in M. We recall a result from linear algebra:

<u>Lemma 3.3:</u> *Let $\nu_1, \nu_2, \ldots, \nu_n$ be n distinct eigenvalues for a linear transformation A and let $y_1, y_2, \ldots, y_n$ be corresponding eigenvectors such that $Ay_i = \nu_i y_i$; $i = 1, 2, \ldots, n$. Suppose M is a linear space invariant under A. If $v = a_1 y_1 + \cdots + a_n y_n \in M$ for $a_1, \ldots, a_n$ nonzero constants, then $y_1, \ldots, y_n \in M$.*

<u>**Proof:**</u> We induct on n, the case $n = 1$ being trivial. So assume the result holds for $n - 1$ distinct eigenvalues.

Let v be as given in the statement. By assumption of invariance, $Av \in M$ and so, by linearity of M, $Av - \nu_n v \in M$. But

$$Av - \nu_n v = a_1(\nu_1 - \nu_n)y_1 + \cdots + a_{n-1}(\nu_{n-1} - \nu_n)y_{n-1} \tag{3.4}$$

and since the ν_i's were distinct, the coefficients on the right side of (3.4) are nonzero and hence by the inductive hypothesis $y_1, \ldots, y_{n-1} \in M$. Thus we see that $y_n \in M$, as required. ∎

Since distinct monomials in M are eigenvectors for $\mathcal{H}$ corresponding to distinct eigenvalues, each monomial occurring in v is also an element of M_0. Then since

$$X_{\alpha_n}^{(\frac{1}{2}\sum a_j)} X_{\alpha_{n-1}}^{(\sum_{j=n-i+1}^{n} a_j)} \cdots X_{\alpha_2}^{(a_{n-1}+a_n)} X_{\alpha_1}^{a_n}(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = c \cdot 1$$

for some nonzero constant $c \in C$ we have that $1 \in M_0$ and hence $M_0 = M$. Thus the module M is an irreducible standard cyclic module of highest weight $-\frac{1}{2}\omega_n$ and having the property that all weight spaces are one-dimensional.

In a similar fashion, if we let N denote the $\mathfrak{U}$-module generated by the monomial x_1, then the set $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \mid a_i \in Z^+; \sum a_i \text{ odd}\}$ forms a basis of N consisting of weight vectors. Arguing exactly as above we have that N is an irreducible standard cyclic module of highest weight $\omega_{n-1} - \frac{3}{2}\omega_n$ having maximal vector x_1 and having one-dimensional weight spaces. Interestingly enough, the modules M and N thus constructed are the only infinite-dimensional irreducible standard cyclic C_n-modules having one-dimensional weight spaces.

Before proving this, two further types of C_n-modules need to be introduced. A C_n-module V is said to be **pointed** if V is irreducible and has a one dimensional weight space. We call a C_n-module V **completely pointed** if V is an infinite dimensional irreducible standard cyclic module in which every weight space is one dimensional. Translated into this terminology, the above claim is that M and N constructed above are (up to equivalence) the only two completely pointed C_n-modules.

We first prove the $n = 2$ case.

<u>Lemma 3.5:</u> *Suppose that $V(\nu)$ is a completely pointed C_2 module. Then* $\nu = -\frac{1}{2}\omega_2$ *or* $\nu = \omega_1 - \frac{3}{2}\omega_2$.

<u>**Proof:**</u> Let $\mathfrak{U}$ be the universal enveloping algebra of C_2 and let $\mathfrak{U}_0$ be the 0-weight space of $\mathfrak{U}$ under the adjoint representation of C_2 on $\mathfrak{U}$. Then as we saw in Section 3.1, $\mathfrak{U}_0$ is a finitely-generated module, generated as an associative algebra by a Cartan basis, along with the basic cycles. As in Section 3.1, we express the basic cycles in terms of a P-B-W basis of $\mathfrak{U}$ so that the Cartan elements are to the right, followed by the positive root vectors, and having the negative root vectors to the left.

i.e. {neg. root vectors} {pos. root vectors} {Cartan elements}

Let v^+ be a maximal vector of V(ν). Since $(V(\nu))_{\nu-m\alpha-n\beta}$ has dimension at most 1, the following pairs of vectors are linearly dependent:

$$Y_{\alpha+\beta}v^+,\ Y_\beta Y_\alpha v^+ \tag{3.6}$$

$$Y_{2\alpha+\beta}v^+,\ Y_\beta Y_\alpha^2 v^+ \tag{3.7}$$

$$Y_{2\alpha+\beta}Y_{\alpha+\beta}Y_\beta v^+,\ Y_{\alpha+\beta}^3 v^+ \tag{3.8}$$

Therefore there exist complex numbers A_1,B_1,A_2,B_2,A_3, and B_3 such that:

$$(A_1 Y_{\alpha+\beta} + B_1 Y_\beta Y_\alpha)\ v^+ = 0 \tag{3.9}$$

with either $A_1 \neq 0$ or $B_1 \neq 0$

$$(A_2 Y_{2\alpha+\beta} + B_2 Y_\beta Y_\alpha^2)\ v^+ = 0 \tag{3.10}$$

with either $A_2 \neq 0$ or $B_2 \neq 0$

$$(A_3 Y_{2\alpha+\beta} Y_{\alpha+\beta} Y_\beta + B_3 Y_{\alpha+\beta}^3)\ v^+ = 0 \tag{3.11}$$

with either $A_3 \neq 0$ or $B_3 \neq 0$

We pull these equations back to the ν-weight space by multiplying:

(3.9) first by $X_{\alpha+\beta}$ and then by $X_\alpha X_\beta$

(3.10) first by $X_{2\alpha+\beta}$ and then by $X_\alpha^2 X_\beta$

and (3.11) first by $X_\beta X_{\alpha+\beta} X_{2\alpha+\beta}$ and then by $X_{\alpha+\beta}^3$.

Now take each of the resulting monomials, for example

$$X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} Y_{2\alpha+\beta} Y_{\alpha+\beta} Y_\beta \ \in \mathfrak{U}_0 \text{ , etc.}$$

and express them as a combination of products of Cartan elements and terms having a positive root vector on the right. In other words, we must move the X's past the Y's. We then let the resulting expressions act on v^+.

For example, operating on (3.11), we have (with $\nu_1 = \nu(H_\alpha)$, $\nu_2 = \nu(H_\beta)$) that

$$
\begin{aligned}
&X_\beta\, X_{\alpha+\beta}\, X_{2\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, X_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta + X_\beta\, X_{\alpha+\beta}\, [X_{2\alpha+\beta}\, Y_{2\alpha+\beta}]\, Y_{\alpha+\beta}\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, X_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta + 4X_\beta\, X_{\alpha+\beta}\, H_\alpha\, Y_{\alpha+\beta}\, Y_\beta \\
&\qquad + 4X_\beta\, X_{\alpha+\beta}\, H_\beta\, Y_{\alpha+\beta}\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, X_{2\alpha+\beta}\, Y_\beta + 2X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, X_\alpha\, Y_\beta \\
&\qquad + 4X_\beta\, H_\alpha\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta \\
&\qquad + 4X_\beta\, H_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta - 4X_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_{2\alpha+\beta} + 2X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_\alpha \\
&\qquad + 4H_\alpha\, X_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta + 8X_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta \\
&\qquad + 4H_\beta\, X_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta - 8X_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta \\
&\qquad - 4X_\beta\, X_{\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_{2\alpha+\beta} + 2X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_\alpha \\
&\qquad + 4(H_\alpha + H_\beta - 1)\, X_\beta\, Y_{\alpha+\beta}\, X_{\alpha+\beta}\, Y_\beta \\
&\qquad + 4(H_\alpha + H_\beta - 1)\, X_\beta\, H_\alpha\, Y_\beta + 8(H_\alpha + H_\beta - 1)\, X_\beta\, H_\beta\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_{2\alpha+\beta} + 2X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_\alpha \\
&\qquad + 4(H_\alpha + H_\beta - 1)\, X_\beta\, Y_{\alpha+\beta}\, Y_\beta\, X_{\alpha+\beta} \\
&\qquad + 4(H_\alpha + H_\beta - 1)\, X_\beta\, Y_{\alpha+\beta}\, X_\alpha \\
&\qquad + 4(H_\alpha + H_\beta - 1)\, H_\alpha\, X_\beta\, Y_\beta + 8(H_\alpha + H_\beta - 1)\, X_\beta\, Y_\beta \\
&\qquad + 8(H_\alpha + H_\beta - 1)\, H_\beta\, X_\beta\, Y_\beta - 16(H_\alpha + H_\beta - 1)\, X_\beta\, Y_\beta \\
&= X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_{2\alpha+\beta} + 2X_\beta\, X_{\alpha+\beta}\, Y_{2\alpha+\beta}\, Y_{\alpha+\beta}\, Y_\beta\, X_\alpha \\
&\qquad + 4(H_\alpha + H_\beta - 1)\, X_\beta\, Y_{\alpha+\beta}\, Y_\beta\, X_{\alpha+\beta}
\end{aligned}
$$

$$+\,4(H_\alpha + H_\beta - 1)\,X_\beta\,Y_{\alpha+\beta}\,X_\alpha$$
$$+\,4(H_\alpha + H_\beta - 1)(H_\alpha + 2H_\beta - 2)\,Y_\beta\,X_\beta\,Y_\beta$$
$$+\,4(H_\alpha + H_\beta - 1)(H_\alpha + 2H_\beta - 2)\,H_\beta \tag{3.12}$$

and

$$\begin{aligned}
&X_\beta\,X_{\alpha+\beta}\,X_{2\alpha+\beta}\,Y^3_{\alpha+\beta}\\
&= X_\beta\,X_{\alpha+\beta}\,Y_{\alpha+\beta}\,X_{2\alpha+\beta}\,Y^2_{\alpha+\beta} + 2X_\beta\,X_{\alpha+\beta}\,X_\alpha\,Y^2_{\alpha+\beta}\\
&= X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_{2\alpha+\beta}\,Y_{\alpha+\beta} + 2X_\beta\,X_{\alpha+\beta}\,Y_{\alpha+\beta}\,X_\alpha\,Y_{\alpha+\beta}\\
&\qquad + 2X_\beta\,X_{\alpha+\beta}\,Y_{\alpha+\beta}\,X_\alpha\,Y_{\alpha+\beta} - 4X_\beta\,X_{\alpha+\beta}\,Y_\beta\,Y_{\alpha+\beta}\\
&= X_\beta\,X_{\alpha+\beta}\,Y^3_{\alpha+\beta}\,X_{2\alpha+\beta} + 2X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha\\
&\qquad + 4X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha - 8X_\beta\,X_{\alpha+\beta}\,Y_{\alpha+\beta}\,Y_\beta\\
&\qquad - 4X_\beta\,Y_\beta\,X_{\alpha+\beta}\,Y_{\alpha+\beta} - 4X_\beta\,X_\alpha\,Y_{\alpha+\beta}\\
&= X_\beta\,X_{\alpha+\beta}\,Y^3_{\alpha+\beta}\,X_{2\alpha+\beta} + 2X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha\\
&\qquad + 4X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha - 8X_\beta\,Y_{\alpha+\beta}\,X_{\alpha+\beta}\,Y_\beta\\
&\qquad - 8X_\beta\,H_\alpha\,Y_\beta - 16X_\beta\,H_\beta\,Y_\beta - 4X_\beta\,Y_\beta\,Y_{\alpha+\beta}\,X_{\alpha+\beta}\\
&\qquad - 4X_\beta\,Y_\beta\,H_\alpha - 8X_\beta\,Y_\beta\,H_\beta - 4X_\beta\,Y_{\alpha+\beta}\,X_\alpha + 8X_\beta\,Y_\beta\\
&= X_\beta\,X_{\alpha+\beta}\,Y^3_{\alpha+\beta}\,X_{2\alpha+\beta} + 2X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha\\
&\qquad + 4X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha - 4X_\beta\,Y_\beta\,Y_{\alpha+\beta}\,X_{\alpha+\beta}\\
&\qquad - 4X_\beta\,Y_{\alpha+\beta}\,X_\alpha - 8X_\beta\,Y_{\alpha+\beta}\,Y_\beta\,X_{\alpha+\beta} - 8X_\beta\,Y_{\alpha+\beta}\,X_\alpha\\
&\qquad - 8H_\alpha\,X_\beta\,Y_\beta - 16X_\beta\,Y_\beta - 16H_\beta\,X_\beta\,Y_\beta + 32X_\beta\,Y_\beta\\
&\qquad - 4Y_\beta\,X_\beta\,H_\alpha - 4H_\beta\,H_\alpha - 8Y_\beta\,X_\beta\,H_\beta - 8H^2_\beta + 8Y_\beta\,X_\beta + 8H_\beta\\
&= X_\beta\,X_{\alpha+\beta}\,Y^3_{\alpha+\beta}\,X_{2\alpha+\beta} + 2X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha\\
&\qquad + 4X_\beta\,X_{\alpha+\beta}\,Y^2_{\alpha+\beta}\,X_\alpha - 4X_\beta\,Y_\beta\,Y_{\alpha+\beta}\,X_{\alpha+\beta}
\end{aligned}$$

$$
\begin{aligned}
&- 4X_\beta\, Y_{\alpha+\beta}\, X_\alpha - 8X_\beta\, Y_{\alpha+\beta}\, Y_\beta\, X_{\alpha+\beta} \\
&- 8X_\beta\, Y_{\alpha+\beta}\, X_\alpha - 12Y_\beta\, X_\beta\, H_\alpha + 8Y_\beta\, X_\beta \\
&- 8(H_\alpha + 2H_\beta - 2)\, Y_\beta\, X_\beta - 12H_\beta(H_\alpha + 2H_\beta - 2).
\end{aligned}
\tag{3.13}
$$

Hence (3.11), (3.12) and (3.13), imply that

$$
\begin{aligned}
0 &= X_\beta\, X_{\alpha+\beta}\, X_{2\alpha+\beta}\,(A_3 Y_{2\alpha+\beta} Y_{\alpha+\beta} Y_\beta + B_3 Y^3_{\alpha+\beta})\, v^+ \\
&= (\nu_1 + 2\nu_2 - 2)\nu_2\big(4(\nu_1 + \nu_2 - 1)A_3 - 12B_3\big)
\end{aligned}
$$

This process gives rise to the following systems of equations:

$$
\left.\begin{aligned}
(\nu_1 + 2\nu_2)\, A_1 + (\nu_1)\, B_1 &= 0 \\
(\nu_1)\, A_1 + (\nu_1\nu_2 + \nu_1)\, B_1 &= 0
\end{aligned}\right\}
\tag{3.14}
$$

$$
\left.\begin{aligned}
4(\nu_1 + 4\nu_2)\, A_2 &= 0 \\
2\nu_1\,(\nu_1 - 1)\,(\nu_2 + 2)\, B_2 &= 0
\end{aligned}\right\}
\tag{3.15}
$$

$$
\left.\begin{aligned}
&4\nu_2\,(\nu_1 + \nu_2 - 1)\,(\nu_1 + 2\nu_2 - 2)\, A_3 \\
&\qquad -12\nu_2\,(\nu_1 + 2\nu_2 - 2)\, B_3 = 0 \\
&\qquad -12\nu_2\,(\nu_1 + 2\nu_2 - 2)\, A_3 \\
&+6\,(\nu_1 + 2\nu_2)\,(\nu_1 + 2\nu_2 - 1)\,(\nu_1 + 2\nu_2 - 2)\, B_3 = 0
\end{aligned}\right\}
\tag{3.16}
$$

Now, each of (3.14), (3.15), and (3.16) has a nontrivial solution and so the corresponding coefficient matrices have determinant = 0. From these equations we get:

$$
\nu_1\nu_2\,(\nu_1 + 2\nu_2 + 2) = 0 \tag{3.17}
$$

$$
8\nu_1\,(\nu_1 - 1)\,(\nu_2 - 2)\,(\nu_1 + \nu_2) = 0 \tag{3.18}
$$

$$(\nu_1 + 2\nu_2 - 2)^2 \left(\nu_2(\nu_1 + \nu_2 - 1)(\nu_1 + 2\nu_2 - 1)(\nu_1 + 2\nu_2) - 6\nu_2^2\right) = 0 \qquad (3.19)$$

Equation (3.17) permits us to study the following cases:

Case 1: $\nu_2 = 0$

Case 2: $\nu_1 = 0$

Case 3: $\nu_1 + 2\nu_2 + 2 = 0$

In Case 1, (3.18) reduces to give

$$16\nu_1^2(\nu_1 - 1) = 0$$

and hence $\nu_1 = 0$ or $\nu_1 = 1$.

In Case 2, since $\nu_1 = 0$, (3.18) gives no information and hence we must use (3.19), which simplifies to give:

$$\begin{aligned} 0 &= (2\nu_2 - 2)^2 \left(\nu_2(\nu_2 - 1)(2\nu_2 - 1)(2\nu_2) - 6\nu_2^2\right) \\ &= 8\nu_2^2(2\nu_2 - 1)(\nu_2 - 2)(\nu_2 - 1)^2 \\ &= 8\nu_2^2(2\nu_2 - 1)(\nu_2 - 2)(\nu_2 - 1)^2 \end{aligned}$$

and hence in this case we get $\nu_1 = 0$ and $\nu_2 = 0, 1, 2, -\frac{1}{2}$.

In Case 3, $\nu_1 = -2(\nu_2 + 1)$ and $\nu_1 \neq 0$, $\nu_2 \neq 0$. In this case (3.19) simplifies to

$$16\left(\nu_2(-\nu_2 - 3)(-3)(-2) - 6\nu_2^2\right) = -96(2\nu_2 + 3) = 0$$

Setting this to 0 implies (since $\nu_2 \neq 0$):

$$\nu_2 = -\frac{3}{2} \, , \, \nu_1 = 1$$

In summary then, the only values of $\nu = \nu_1\omega_1' + \nu_2\omega_2'$ which can satisfy the hypotheses are:

(*i*) $\nu_2 = 0$; $\nu_1 = 0, 1$

(*ii*) $\nu_1 = 0$; $\nu_2 = 1, 2, -\frac{1}{2}$

(*iii*) $\nu_1 = 1$; $\nu_2 = -\frac{3}{2}$.

Now, $\nu = 0$, $\nu = \omega_1$, $\nu = \omega_2$, and $\nu = 2\omega_2$ can all be discarded since they are dominant integral and hence the corresponding simple highest weight module is finite dimensional. The remaining two possibilities are $\nu = -\frac{1}{2}\omega_2$ and $\nu = \omega_1 - \frac{3}{2}\omega_2$ as stated in the proposition. Since these last two possibilities are not dominant integral, the corresponding simple highest weight modules are infinite dimensional. ∎

This result may be extended to C_n.

__Proposition 3.20:__ *Suppose that $V(\nu)$ is a completely pointed C_n-module. Then either $\nu = -\frac{1}{2}\omega_n$ or $\nu = \omega_{n-1} - \frac{3}{2}\omega_n$.*

__Proof:__ We induct on n, the case $n = 2$ being done above.

Suppose that $V(\nu)$ is a completely pointed C_n-module, with

$$\nu = \sum_{i=1}^{n} b_i\omega_i.$$

Some C_2-subalgebras of C_n are described by the following root subsystems:

$$\{\alpha_i + \cdots + \alpha_{n-1}, \alpha_n\} \qquad i = 1, 2, \ldots, n-1$$

$$\{\alpha_i + \cdots + \alpha_j, 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n\} \qquad 1 \leq i \leq j \leq n.$$

By considering the corresponding sets of fundamental dominant weights for each root subsystem, we get the following sets of coefficients:

$$\{ b_i + \cdots + b_{n-1}, b_n \} \qquad i = 1, 2, \ldots, n-1 \tag{3.21}$$

$$\{ b_i + \cdots + b_j, b_{j+1} + \cdots + b_n \} \qquad 1 \leq i \leq j \leq n. \tag{3.22}$$

Now, using the root subsystem $\{ \alpha_{n-1}, \alpha_n \}$, by the analysis in Lemma 3.6, using only the one-dimensionality of the weight spaces, we must have $b_n = -\frac{3}{2}$, $-\frac{1}{2}$,1, or 0.

If $b_n = -\frac{3}{2}$, then using (3.21), we have

$$b_i + \cdots + b_{n-1} = 1 \qquad \text{for all } i = 1, \ldots, n-1$$

and hence $b_{n-1} = 1$ and $b_i = 0$ for $i = 1, \ldots, n-2$.

If $b_n = -\frac{1}{2}$ then (3.21) implies that

$$b_i + \cdots + b_{n-1} = 0 \qquad \text{for all } i = 1, \ldots, n-1$$

and hence $b_i = 0$ for all $i = 1, \ldots, n-1$.

If $b_n = 1$ then (3.21) implies that

$$b_i + \cdots + b_{n-1} = 0 \qquad \text{for all } i = 1, \ldots, n-1$$

and hence $b_i = 0$ for all $i = 1, \ldots, n-1$.

Finally, if $b_n = 0$, then by (3.21),

$$b_i + \cdots + b_{n-1} = 0 \text{ or } 1 \qquad \text{for each } i = 1, \ldots, n-1$$

If $b_j = 0$ for all $j = 1, \ldots, n$, then $\nu = 0$. Otherwise, let j be maximal such that $b_j = 1$. Using (3.22) we see that $b_i = 0$ for $1 \leq i < j$.

To summarize, we have the following possibilities:

$$(a)\quad b_n = -\frac{1}{2}, b_i = 0 \quad i = 1, \ldots, n-1$$

$$(b)\quad b_{n-1} = 1, b_n = -\frac{3}{2}, b_i = 0 \quad i = 1, \ldots, n-2$$

$$(c)\quad b_i = 0 \quad i = 1, \ldots, n$$

$$(d)\quad b_j = 1, b_i = 0 \quad i \neq j \text{ for } j = 1, \ldots, n.$$

But the irreducible standard cyclic modules having highest weight 0 and ω_j for $j = 1, \ldots, n$ are finite dimensional. Thus the only possibilities for ν for which $V(\nu)$ is completely pointed are the two stated. ∎

Thus, up to isomorphism, M and N as constructed previously are the only completely pointed C_n-modules. In the remainder of this chapter, we let

$$\nu = -\frac{1}{2}\omega_n \qquad \text{and} \qquad \nu' = \omega_{n-1} - \frac{3}{2}\omega_n\,, \tag{3.23}$$

so that ν and ν' are the highest weights of M and N, respectively.

3.3 Tensoring M and N with a Finite Dimensional Module

Keeping the notation as in Sections 3.1 and 3.2, we also let $\lambda \in P^+$, so that in particular $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$. As before, we let $V = V(\lambda)$ be the irreducible highest weight module having highest weight λ and fixed maximal vector v^+. Recall that since $\lambda \in P^+$, $V(\lambda)$ is finite dimensional. Since $V = \bigoplus \sum V_\mu$, a direct sum of weight spaces, we may choose a basis $\{v_0, v_1, \ldots, v_m\}$ of V where the v_i are weight vectors and $v_0 = v^+$. We wish to consider the modules $M \otimes V$ and $N \otimes V$. We have that

$$M \otimes V = \mathrm{sp}\left\{x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n} \otimes v_i \mid \sum_{j=1}^{n} a_j \text{ even}; a_j \in Z^+; 0 \leq i \leq m\right\}$$

and

$$N \otimes V = \mathrm{sp}\,\{x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \otimes v_i \mid \sum_{j=1}^{n} a_j \text{ odd}; a_j \in Z^+; 0 \leq i \leq m\}.$$

For later use we need to calculate the vector δ in terms of the ε-basis of $\mathbb{R}^n$. Recall that $\delta = \sum_{j=1}^{n} \omega_j$ so that, using (3.1), we have:

$$\delta = -\left(\sum_{k=1}^{n} k\varepsilon_k\right). \tag{3.24}$$

Also, for the module $M \otimes V$, we define the function wt on all weight vectors of $M \otimes V$ by setting $\mathrm{wt}\, v = \mu$ if and only if $v \in V_\mu$.

<u>Lemma 3.25:</u> *Let*

$$w^+ = \sum_{a_1, \ldots, a_n, i} c_{a_1 \ldots a_n i}\, x_1^{a_1} \ldots x_n^{a_n} \otimes v_i \tag{3.26}$$

be a nonzero maximal vector of weight μ occurring in $M \otimes V$, where the sum is over all distinct monomials $x_1^{a_1} \ldots x_n^{a_n} \otimes v_i$ with $wt(x_1^{a_1} \ldots x_n^{a_n} \otimes v_i) = \mu$. Then there exists an index i_0 such that $c_{0 \ldots 0 i_0} \neq 0$.

<u>Proof:</u> Since w^+ is nonzero, we may let a_n^0 be minimal among the a_n such that for some choice of $a_1, \ldots, a_{n-1}$, and i, $c_{a_1 \ldots a_{n-1} a_n^0 i} \neq 0$. Then

$$\begin{aligned}
0 = X_{\alpha_1}(w^+) = &\sum_{\substack{a_n^0 < a_n \\ a_1, \ldots, a_{n-1}, i}} a_n\, c_{a_1 \ldots a_n i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}+1} x_n^{a_n - 1} \otimes v_i \\
&+ \sum_{\substack{a_n^0 < a_n \\ a_1, \ldots, a_{n-1}, i}} c_{a_1 \ldots a_n i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n} \otimes X_{\alpha_1} \cdot v_i \\
&+ \sum_{a_1, \ldots, a_{n-1}, i} a_n^0\, c_{a_1 \ldots a_{n-1} a_n^0 i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}+1} x_n^{a_n^0 - 1} \otimes v_i \\
&+ \sum_{a_1, \ldots, a_{n-1}, i} c_{a_1 \ldots a_{n-1} a_n^0 i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n^0} \otimes X_{\alpha_1} \cdot v_i
\end{aligned}$$

By the minimality of a_n^0,

$$\sum_{a_1,\ldots,a_{n-1},i} a_n^0\, c_{a_1\ldots a_{n-1}i}\, x_1^{a_1}\ldots x_{n-1}^{a_{n-1}+1} x_n^{a_n^0-1}\otimes v_i = 0\ .$$

If $a_n^0 \neq 0$, then by linear independence of the basis vectors, we have $c_{a_1\ldots a_{n-1}a_n^0 i} = 0$ for every choice of $a_1,\ldots,a_{n-1},i$, contrary to the choice of a_n^0. Hence there exist $a_1,\ldots,a_{n-1},i$ such that $c_{a_1\ldots a_{n-1}0i} \neq 0$.

This argument may be repeated, using $X_{\alpha_2}, X_{\alpha_3},\ldots,X_{\alpha_{n-1}}$, to give the existence of a_1, i such that $c_{a_1 0\ldots 0i} \neq 0$.

Finally, choose a_1^0 minimal such that $c_{a_1^0 0\ldots 0i} \neq 0$ for some choice of i. By applying X_{α_n} to w^+ as above, we get that

$$\sum_i -\frac{1}{2}a_1^0(a_1^0-1)\, c_{a_1^0 0\ldots 0i}\, x_1^{a_1^0-2}\otimes v_i = 0$$

Since $x_1^{a_1^0-2}$ is a monomial belonging to M, we must have a_1^0 even. In particular, $a_1^0 - 1 \neq 0$. If $a_1^0 \neq 0$ then $a_1^0(a_1^0-1)\neq 0$ and the argument used previously again gives $c_{a_1^0 0\ldots 0i} = 0$ for all choices of i, contrary to the choice of a_1^0. Thus $a_1^0 = 0$ and hence there is an i_0 such that $c_{0\ldots 0i_0} \neq 0$. ∎

A similar result holds in $N\otimes V$:

Lemma 3.27: *Let w^+ be a nonzero maximal vector (as in (3.26)) occurring in $N\otimes V$, where the sum is over all distinct monomials $x_1^{a_1}\ldots x_n^{a_n}\otimes v_i$ with $wt_{N\otimes V}(x_1^{a_1}\ldots x_n^{a_n}\otimes v_i) = \mu$. Then there exists an index i_0 such that $c_{10\ldots 0i_0} \neq 0$.*

Proof: The proof is identical to that of Lemma 3.25 except that the minimal a_1^0 chosen must in this case be odd and hence $a_1^0 = 1$. ∎

Lemma 3.28: *Let w^+ be a nonzero maximal vector (as in (3.26)) occurring in $M\otimes V$ or $N\otimes V$. Then there exist indices $a_1^0, a_2^0,\ldots,a_n^0$ such that $c_{a_1^0\ldots a_n^0 0} \neq 0$.*

Proof: Since w^+ is nonzero, we may choose a_n^0 maximal such that for some choice of $a_1, \ldots, a_{n-1}, i$, we have $c_{a_1 \ldots a_{n-1} a_n^0 i} \neq 0$. Then

$$
\begin{aligned}
0 = X_{\alpha_1}(w^+) \\
= \sum_{\substack{a_n < a_n^0 \\ a_1, \ldots, a_{n-1}, i}} & a_n\, c_{a_1 \ldots a_n i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}+1} x_n^{a_n - 1} \otimes v_i \\
+ \sum_{\substack{a_n < a_n^0 \\ a_1, \ldots, a_{n-1}, i}} & c_{a_1 \ldots a_n i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n} \otimes X_{\alpha_1} \cdot v_i \\
+ \sum_{a_1, \ldots, a_{n-1}, i} & a_n^0\, c_{a_1 \ldots a_{n-1} a_n^0 i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}+1} x_n^{a_n^0 - 1} \otimes v_i \\
+ \sum_{a_1, \ldots, a_{n-1}, i} & c_{a_1 \ldots a_{n-1} a_n^0 i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n^0} \otimes X_{\alpha_1} \cdot v_i
\end{aligned}
$$

By maximality of a_n^0,

$$
\sum_{a_1, \ldots, a_{n-1}, i} c_{a_1 \ldots a_{n-1} a_n^0 i}\, x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n^0} \otimes X_{\alpha_1} \cdot v_i = 0 \tag{3.29}
$$

But then (3.29) becomes:

$$
\sum_{a_1, \ldots, a_{n-1}} x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n^0} \otimes X_{\alpha_1}\Big(\sum_i c_{a_1 \ldots a_{n-1} a_n^0 i}\, v_i\Big) = 0 \tag{3.30}
$$

Moreover, since the monomials are distinct, the terms in the sum in (3.30) are linearly independent. Hence for each distinct set of values $\{a_1, \ldots, a_{n-1}\}$,

$$
x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} x_n^{a_n^0} \otimes X_{\alpha_1}\Big(\sum_i c_{a_1 \ldots a_{n-1} a_n^0 i}\, v_i\Big) = 0
$$

and so

$$
X_{\alpha_1}\Big(\sum_i c_{a_1 \ldots a_{n-1} a_n^0 i}\, v_i\Big) = 0 \tag{3.31}
$$

Now let a_{n-1}^0 be maximal such that for some $a_1, \ldots, a_{n-2}$, and i, we have $c_{a_1 \ldots a_{n-2} a_{n-1}^0 a_n^0 i} \neq 0$. Then

$$
X_{\alpha_i}\Big(\sum_i c_{a_1 \ldots a_{n-2} a_{n-1}^0 a_n^0 i}\, v_i\Big) = 0
$$

and since the v_i's form a basis for V,

$$\sum_i c_{a_1 \dots a_{n-2} a^0_{n-1} a^0_n i}\, v_i \neq 0 .$$

Now

$$\begin{aligned}
0 &= X_{\alpha_2}(w^+) \\
&= \sum_{\substack{a_{n-1} < a^0_{n-1} \\ a_1,\dots,a_{n-2},a_n,i}} X_{\alpha_2}(c_{a_1 \dots a_{n-2} a_{n-1} a_n i}\, x_1^{a_1} \dots x_{n-1}^{a_{n-1}} x_n^{a_n} \otimes v_i) \\
&+ \sum_{\substack{a_n < a^0_n \\ a_1,\dots,a_{n-2},i}} X_{\alpha_2}(c_{a_1 \dots a_{n-2} a^0_{n-1} a_n i}\, x_1^{a_1} \dots x_{n-1}^{a^0_{n-1}} x_n^{a_n} \otimes v_i) \\
&+ \sum_{a_1,\dots,a_{n-2},i} a^0_{n-1}\, c_{a_1 \dots a^0_{n-1} a^0_n i}\, x_1^{a_1} \dots x_{n-2}^{a_{n-2}+1} x_{n-1}^{a^0_{n-1}-1} x_n^{a^0_n} \otimes v_i \\
&+ \sum_{a_1,\dots,a_{n-2},i} c_{a_1 \dots a_{n-2} a^0_{n-1} a^0_n i}\, x_1^{a_1} \dots x_{n-2}^{a_{n-2}} x_{n-1}^{a^0_{n-1}} x_n^{a^0_n} \otimes X_{\alpha_2}(v_i)
\end{aligned}$$

By maximality of a^0_{n-1},

$$\begin{aligned}
0 &= \sum_{a_1,\dots,a_{n-2},i} c_{a_1 \dots a_{n-2} a^0_{n-1} a^0_n i}\, x_1^{a_1} \dots x_{n-2}^{a_{n-2}} x_{n-1}^{a^0_{n-1}} x_n^{a^0_n} \otimes X_{\alpha_2}(v_i) \\
&= \sum_{a_1,\dots,a_{n-2}} \Big[x_1^{a_1} \dots x_{n-2}^{a_{n-2}} x_{n-1}^{a^0_{n-1}} x_n^{a^0_n} \\
&\qquad\qquad \otimes X_{\alpha_2}\Big(\sum_i c_{a_1 \dots a_{n-2} a^0_{n-1} a^0_n i}\, v_i\Big)\Big] \qquad (3.32)
\end{aligned}$$

and again by linear independence of the terms in the sum (3.32), for each distinct set of values $\{a_1, \dots, a_{n-2}\}$ we have

$$X_{\alpha_2}\Big(\sum_i c_{a_1 \dots a_{n-2} a^0_{n-1} a^0_n i}\, v_i\Big) = 0 .$$

Applying a similar argument to $a_{n-2}, a_{n-3}, \dots, a_2$, we have maximal indices

$a^0_{n-2}, \ldots, a^0_2$ with the property that there are a_1 and i such that

$$\sum_i c_{a_1 a^0_2 \ldots a^0_{n-1} a^0_n i}\, v_i \neq 0$$

$$\text{and} \qquad X_{\alpha_k}\left(\sum_i c_{a_1 a^0_2 \ldots a^0_n i}\, v_i\right) = 0 \qquad (1 \leq k \leq n-1)$$

We may then choose a^0_1 maximal such that for some i,

$$\sum_i c_{a^0_1 a^0_2 \ldots a^0_{n-1} a^0_n i}\, v_i \neq 0 \tag{3.33}$$

$$\text{and} \qquad X_{\alpha_k}\left(\sum_i c_{a^0_1 a^0_2 \ldots a^0_n i}\, v_i\right) = 0 \qquad (1 \leq k \leq n-1) \tag{3.34}$$

Then

$$\begin{aligned}
0 &= X_{\alpha_n}(w^+) \\
&= \sum_i \sum_{(a_1,\ldots,a_n) \neq (a^0_1,\ldots,a^0_n)} X_{\alpha_n}(c_{a_1 \ldots a_n i}\, x_1^{a_1} \ldots x_n^{a_n} \otimes v_i) \\
&+ \sum_i (-\frac{1}{2}) a^0_1 (a^0_1 - 1)\, c_{a^0_1 \ldots a^0_n i}\, x_1^{a^0_1 - 2} \ldots x_n^{a^0_n} \otimes v_i \\
&+ \sum_i c_{a^0_1 \ldots a^0_n i}\, x_1^{a^0_1} \ldots x_n^{a^0_n} \otimes X_{\alpha_n}(v_i)\ .
\end{aligned}$$

Repeating again the argument used above, we see that

$$X_{\alpha_n}\left(\sum_i c_{a^0_1 \ldots a^0_n i}\, v_i\right) = 0\ . \tag{3.35}$$

Combining equations (3.33), (3.34), and (3.35) we conclude that $\sum_i c_{a^0_1 \ldots a^0_n i}\, v_i$ is a nonzero maximal vector in $V(\lambda)$. But since $V(\lambda)$ is irreducible, we have for some nonzero constant c,

$$\sum_i c_{a^0_1 \ldots a^0_n i}\, v_i = c v^+\ .$$

In particular, since $\{v^+ = v_0, v_1, \ldots, v_n\}$ is a basis for $V(\lambda)$, we must have $c_{a^0_1 \ldots a^0_n 0} = c \neq 0$. ∎

3. Completely Pointed C_n-Modules

Recall from Section 2.2 that $\Pi(\lambda)$ denotes the set of weight functions of $V(\lambda)$.

<u>Lemma 3.36:</u> *Let $\theta \in \Phi^+$ and let $S = \{\mu, \mu - \theta, \ldots, \mu - p\theta\}$ be a θ-string of weights of $V = V(\lambda)$ such that*

$$(a)\ \ \mu + \theta, \mu - (p+1)\theta \notin \Pi(\lambda)$$

$$(b)\ \ \text{for each } \nu \in S,\ \dim V(\lambda)_\nu = 1\ .$$

Also, let u_i be the corresponding basis vector in $V(\lambda)_{\mu - i\theta}$ for $0 \leq i \leq p$. Suppose $w = \sum_{i=1}^{p} A_i x_1^{a_{1i}} \cdots x_n^{a_{ni}} \otimes u_i$ is a vector in a weight space of $M \otimes V$ or $N \otimes V$ such that $X_\theta(w) = 0$. If $w \neq 0$ and if q is maximum such that $A_q \neq 0$, then $X_\theta(x_1^{a_{1q}} \cdots x_n^{a_{nq}}) = 0$.

<u>**Proof:**</u> We prove the contrapositive. Suppose then that q is maximum with $A_q \neq 0$ and yet $X_\theta(x_1^{a_{1q}} \cdots x_n^{a_{nq}}) \neq 0$.

First, since each V_ν ($\nu \in S$) is 1-dimensional, we have, for $i = 1, \ldots, p$, $X_\theta(u_i) = c_i u_{i-1}$ for some constant $c_i \in \mathbb{C}$. Also, $X_\theta(u_0) \in V_{\mu+\theta} = (0)$ and so $X_\theta(u_0) = 0$. Applying X_θ to w we have:

$$\begin{aligned} 0 &= X_\theta(w) \\ &= \sum_{i=0}^{q} A_i X_\theta(x_1^{a_{1i}} \cdots x_n^{a_{ni}}) \otimes u_i + \sum_{i=0}^{q} A_i x_1^{a_{1i}} \cdots x_n^{a_{ni}} \otimes X_\theta(u_i). \end{aligned}$$

In particular, no vector of the form $x_1^{a_{1i}} \cdots x_n^{a_{ni}} \otimes X_\theta(u_i)$ can equal $x_1^{a_{1q}} \cdots x_n^{a_{nq}} \otimes u_q$. By linear independence of the vectors, then, $A_q X_\theta(x_1^{a_{1q}} \cdots x_n^{a_{nq}}) \otimes u_q = 0$. Since $X_\theta(x_1^{a_{1q}} \cdots x_n^{a_{nq}}) \neq 0$, we must have $A_q = 0$, contrary to the choice of q. ∎

<u>**Remark:**</u> If $V(\lambda)$, with $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_n$, is any finite dimensional C_n-module (so each $\lambda_k \in \mathbb{Z}^+$), then for any $i = 1, \ldots, n$, Lemma 3.36 applies with $\mu = \lambda$, $\theta = \alpha_i$, and $p = \lambda_i$ (ie. the Lemma applies to the α_i-string through λ). In particular, if we take w^+ to be a nonzero maximal vector in $M \otimes V(\lambda)$ or $N \otimes V(\lambda)$

having highest weight μ_0, then by Lemma 3.28, $x_1^{a'_1}\cdots x_n^{a'_n}\otimes v^+$ occurs in w with nonzero coefficient. If we choose q maximal such that $A_q \neq 0$ (as in Lemma 3.36), with corresponding monomial $x_1^{a_{1t}}\cdots x_n^{a_{nt}}$, then $X_{\alpha_i}^{q}(x_1^{a'_1}\cdots x_n^{a'_n}) = c x_1^{a_{1t}}\cdots x_n^{a_{nt}}$, and so Lemma 3.36 implies that $X_{\alpha_i}^{q+1}(x_1^{a'_1}\cdots x_n^{a'_n}) = 0$. Since $q \leq \lambda_i$, we have in particular that $X_{\alpha_i}^{\lambda_i+1}(x_1^{a'_1}\cdots x_n^{a'_n}) = 0$. In other words, we must have:

$$(x_{n-i}\partial_{n-i+1})^{\lambda_i+1}(x_1^{a'_1}\cdots x_n^{a'_n}) = 0 \qquad 1 \leq i \leq n-1;$$

and

$$(-\frac{1}{2}\partial_1^2)^{\lambda_n+1}(x_1^{a'_1}\cdots x_n^{a'_n}) = 0$$

which implies that $0 \leq a'_1 \leq 2\lambda_n + 1$, and for each $i = 2,\ldots,n$, we must have $0 \leq a'_{n-i+1} \leq \lambda_i$. This proves:

Proposition 3.37: *If w^+ is a nonzero maximal vector in $M\otimes V$ or $N \otimes V$ (as in (3.26)) and $a'_1,\ldots,a'_n$ are such that $c_{a'_1\cdots a'_n 0} \neq 0$ then the a'_i must satisfy*

$$\begin{aligned} &0 \leq a'_1 \leq 2\lambda_n + 1 \\ &0 \leq a'_i \leq \lambda_{n-i+1} \qquad (2 \leq i \leq n). \ \blacksquare \end{aligned} \tag{3.38}$$

Proposition 3.39: *Let $V = V(\lambda)$ with λ a dominant integral weight. Let w_1^+ and w_2^+ be maximal vectors in $M\otimes V$ having weights μ_1 and μ_2, respectively. If μ_1 is linked to μ_2 then $\mu_1 = \mu_2$.*

Proof: Let w_1^+ and μ_1 be as given. We first find conditions on the coefficients of $\mu_1 + \delta$ in the expression

$$\mu_1 + \delta = d_1\epsilon_1 + \cdots + d_n\epsilon_n\ . \tag{3.40}$$

Express w_1^+ as in (3.26) and write $\lambda = \lambda_1\omega_1 + \cdots + \lambda_n\omega_1$. By Lemma 3.28 and Proposition 3.37 we have a term of the form $x_1^{a'_1}\cdots x_n^{a'_n}\otimes v^+$ in the expression for

w_1^+ with nonzero coefficient, where $\sum_{j=1}^n a'_j$ is even and equations (3.38) hold. By Propositions 3.25, 3.28 and 3.37, we must have summands of the form $x_1^{a'_1} \cdots x_n^{a'_n} \otimes v^+$ and $1 \otimes v$ with nonzero coefficients in W_1^+, where $v \in V_{\mu_0}$ for some weight μ_0 of V. Hence

$$wt(1 \otimes v) = wt(x_1^{a'_1} \cdots x_n^{a'_n} \otimes v^+)$$

and hence

$$wt(1) + wt(v) = wt(x_1^{a'_1} \cdots x_n^{a'_n}) + wt(v^+).$$

This implies that

$$\begin{aligned}
\mu_0 &= \frac{1}{2}\omega_n + (a'_{n-1} - a'_n)\omega_1 + \cdots + (a'_1 - a'_2)\omega_{n-1} - (a'_1 + \frac{1}{2})\omega_n \\
&\quad + \lambda_1\omega_1 + \cdots + \lambda_n\omega_n \\
&= (\lambda_1 + a'_{n-1} - a'_n)(-\varepsilon_n) + (\lambda_2 + a'_{n-2} - a'_{n-1})(-\varepsilon_{n-1} - \varepsilon_n) \\
&\quad + (\lambda_3 + a'_{n-3} - a'_{n-2})(-\varepsilon_{n-2} - \varepsilon_{n-1} - \varepsilon_n) + \cdots \\
&\quad + (\lambda_i + a'_{n-i} - a'_{n-i+1})(-\varepsilon_{n-i+1} - \varepsilon_{n-i+2} - \cdots - \varepsilon_n) + \cdots \\
&\quad + (\lambda_{n-1} + a'_1 - a'_2)(-\varepsilon_2 - \cdots - \varepsilon_n) \\
&\quad + (\lambda_n - a'_1)(-\varepsilon_1 - \cdots - \varepsilon_n) \\
&= (a'_1 - \lambda_n)\varepsilon_1 + (a'_2 - \lambda_{n-1} - \lambda_n)\varepsilon_2 \\
&\quad + \cdots + (a'_i - \lambda_{n-i+1} - \lambda_{n-i+2} - \cdots - \lambda_n)\varepsilon_i + \cdots \\
&\quad + (a'_{n-1} - \lambda_2 - \cdots - \lambda_n)\varepsilon_{n-1} + (a'_n - \lambda_1 - \cdots - \lambda_n)\varepsilon_n
\end{aligned}$$

Now, from (3.24) we have, since $\mu_1 = \nu + \mu_0$,

$$\begin{aligned}
\mu_1 + \delta &= \nu + \mu_0 + \delta \\
&= \frac{1}{2}\varepsilon_1 + \cdots + \frac{1}{2}\varepsilon_n - \varepsilon_1 - 2\varepsilon_2 - \cdots - n\varepsilon_n
\end{aligned}$$

$$+ (a'_1 - \lambda_n)\epsilon_1 + (a'_2 - \lambda_{n-1} - \lambda_n)\epsilon_2$$
$$+ \cdots + (a'_i - \lambda_{n-i+1} - \lambda_{n-i+2} - \cdots - \lambda_n)\epsilon_i + \cdots$$
$$+ (a'_{n-1} - \lambda_2 - \cdots - \lambda_n)\epsilon_{n-1} + (a'_n - \lambda_1 - \cdots - \lambda_n)\epsilon_n$$
$$= \left(a'_1 - \lambda_n - \frac{1}{2}\right)\epsilon_1 + \left(a'_2 - \lambda_{n-1} - \lambda_n - \frac{3}{2}\right)\epsilon_2$$
$$+ \left(a'_3 - \lambda_{n-2} - \lambda_{n-1} - \lambda_n - \frac{5}{2}\right)\epsilon_3 + \cdots$$
$$+ \left(a'_i - \lambda_{n-i+1} - \lambda_{n-i+2} - \cdots - \lambda_n - \frac{(2i-1)}{2}\right)\epsilon_i + \cdots$$
$$+ \left(a'_n - \lambda_1 - \lambda_2 - \cdots - \lambda_n - \frac{(2n-1)}{2}\right)\epsilon_n \tag{3.41}$$

Using equations (3.38) we have conditions on the coefficients d_i in equation (3.40) as follows:

$$-(\lambda_n + \frac{1}{2}) \le d_1 \le (\lambda_n + \frac{1}{2})$$
$$-(\lambda_{n-1} + \lambda_n + \frac{3}{2}) \le d_2 \le -(\lambda_n + \frac{3}{2})$$
$$\vdots$$
$$-(\lambda_{n-i+1} + \cdots + \lambda_n + \frac{(2i-1)}{2}) \le$$
$$d_i \le -(\lambda_{n-i+2} + \cdots + \lambda_n + \frac{(2i-1)}{2})$$
$$\vdots$$
$$-(\lambda_1 + \cdots \lambda_n + \frac{(2n-1)}{2} \le d_n \le -(\lambda_2 + \cdots + \lambda_n + \frac{(2n-1)}{2})$$

and hence we have

$$d_n < d_{n-1} < \cdots < d_2 < -(\lambda_n + \frac{1}{2}) \le d_1 \le (\lambda_n + \frac{1}{2}) \tag{3.42}$$

To complete the proof, let w_1^+, w_2^+, μ_1, and μ_2 be as given in the statement of the proposition. We then have

$$\mu_1 + \delta = d'_1\epsilon_1 + \cdots + d'_n\epsilon_n$$
$$\mu_2 + \delta = d''_1\epsilon_1 + \cdots + d''_n\epsilon_n$$

where

$$d'_n < d'_{n-1} < \cdots < d'_2 < -(\lambda_n + \frac{1}{2}) \leq d'_1 \leq (\lambda_n + \frac{1}{2})$$
$$d''_n < d''_{n-1} < \cdots < d''_2 < -(\lambda_n + \frac{1}{2}) \leq d''_1 \leq (\lambda_n + \frac{1}{2})$$

Note in particular that $d'_j < 0$ and $d''_j < 0$ for $2 \leq j \leq n$. Hence if $\sigma \in \mathcal{W}$ is such that $\sigma(\mu_1 + \delta) = \mu_2 + \delta$, then necessarily we must have

$$\sigma(\epsilon_j) = \epsilon_j \qquad (2 \leq j \leq n)$$

Now if $\sigma(\epsilon_1) = -\epsilon_1$ then $d''_1 = -d'_1$, which implies, by (3.41), that

$$a''_1 = 2\lambda_n + 1 - a'_1$$

Also, for $2 \leq j \leq n$ we have $d'_j = d''_j$, and hence $a'_j = a''_j$. Therefore

$$\sum_{k=1}^{n} a''_k = \sum_{k=1}^{n} a'_k - 2a'_1 + 2\lambda_n + 1 \tag{3.43}$$

But $\sum_{k=1}^{n} a''_k$ and $\sum_{k=1}^{n} a'_k$ are even, contradicting equation (3.43). Thus $\sigma =$ id and $\mu_1 = \mu_2$. ∎

A similar argument gives

Proposition 3.44: *Let $V = V(\lambda)$ with λ a dominant integral weight. Let w_1^+ and w_2^+ be maximal vectors in $N \otimes V$ having weights μ_1 and μ_2, respectively. If μ_1 is linked to μ_2 then $\mu_1 = \mu_2$.* ∎

Proposition 3.45: *Let $V = V(\lambda)$ with λ a dominant integral weight. Let w^+ be a nonzero maximal vector in $M \otimes V$ or $N \otimes V$. Then the standard cyclic module $W = \mathfrak{U} \cdot w^+$ is irreducible.*

Proof: We prove the case $M \otimes V$, the case $N \otimes V$ being similar. Let μ denote the weight of w^+.

Suppose that Y is a nonzero submodule of W. Then Y is standard cyclic and hence contains a nonzero maximal vector y^+, say of highest weight μ'. Since W admits exactly one central character, we must have $\chi_\mu = \chi_{\mu'}$, i.e. μ is linked to μ'. Clearly w^+, y^+ are nonzero maximal vectors in $M \otimes V$, and hence by Proposition 3.39, $\mu = \mu'$, i.e. $Y = W$.

Thus W is irreducible. ∎

__Theorem 3.46:__ *Let $V = V(\lambda)$ be a finite dimensional irreducible C_n-module. The conditions in (3.38), along with the parity of $\sum_{i=1}^n a_i$, give a complete listing of all possible maximal vectors which can occur in $M \otimes V$ and $N \otimes V$, and hence all the possible standard cyclic submodules of $M \otimes V$ and $N \otimes V$. In particular, each such standard cyclic submodule occurs at most once. Furthermore, these conditions classify all the possible central characters which can occur.*

__Proof:__ The first statement follows directly from Proposition 3.37, since any standard cyclic submodule must have a maximal vector. We prove the other two statements for $M \otimes V$, the case $N \otimes V$ being similar.

To see that each standard cyclic submodule occurs at most once, let $M_1 \cong M_2$ be standard cyclic submodules of highest weight μ with corresponding maximal vectors w_1^+ and w_2^+. Suppose that $M_1 \neq M_2$. Then clearly w_1^+ and w_2^+ are nonproportional. Using Lemma 3.28, we may write

$$w_1^+ = a x_1^{a_1} \cdots x_n^{a_n} \otimes v^+ + \cdots$$

$$w_2^+ = b x_1^{a_1} \cdots x_n^{a_n} \otimes v^+ + \cdots$$

where $a, b \neq 0$. Then $w = b w_1^+ - a w_2^+ \neq 0$ is a maximal vector of highest weight μ. But the expression for w in terms of basis elements does not contain a nonzero multiple of $x_1^{a_1} \cdots x_n^{a_n} \otimes v^+$, contrary to Lemma 3.28. Hence $w_1^+ = w_2^+$ and $M_1 = M_2$.

Let $(a_1^{(1)}, \ldots, a_n^{(1)}), \ldots, (a_1^{(p)}, \ldots, a_n^{(p)})$ be a complete listing of all n-tuples satisfying (3.38) and such that $\sum_{i=1}^n a_i^{(j)}$ even. For each $j = 1, \ldots, p$, let ν_i be the weight of the monomial $x_1^{a_1^{(j)}} \cdots x_n^{a_n^{(j)}} \otimes v^+$, i.e. the highest weight corresponding to the n-tuple $(a_1^{(j)}, \ldots, a_n^{(j)})$. As usual, let χ_{ν_j} denote the central character admitted by the irreducible standard cyclic module of highest weight ν_j. We wish to show that no other central characters may occur in $M \otimes V$.

As in Section 2.6, we can decompose $M \otimes V$ as a direct sum of generalized eigenspaces belonging to distinct central characters of $M \otimes V$:

$$M \otimes V = M^{(1)} \oplus M^{(2)} \oplus \cdots \oplus M^{(k)}.$$

Let $\chi^{(i)}$ be the central character corresponding to $M^{(i)}$. Each $M^{(i)}$ inherits a filtration from $M \otimes V$:

$$M^{(i)} = M_n^{(i)} \supsetneq M_{n-1}^{(i)} \supsetneq \cdots \supsetneq M_1^{(i)} \supsetneq M_0^{(i)} = (0).$$

where each quotient is an irreducible highest weight module. Consider the last nonzero factor $M_1^{(i)}$ in this filtration. $M_1^{(i)}$ is an irreducible standard cyclic submodule of M_i, and hence of $M \otimes V$, and must contain a nonzero maximal vector w^+. But the equations in (3.38) classify all such possible w^+. Hence $\chi^{(i)} = \chi_{\nu_j}$ for some $1 \leq j \leq p$.

Thus the central characters χ_{ν_j} for $j = 1, \ldots, p$ are the only central characters which may occur in $M \otimes V$. ∎

Chapter 4

Conclusion

At this point, it is useful to review what we have accomplished in this work.

Since the publication of Fernando's work ([F]), the problem of determining all simple torsion free A_n- and C_n-modules has become very important. Britten and Lemire ([BL]) classified all pointed torsion free modules for A_n and C_n, and the work of Britten, Futorny and Lemire ([BFL]) showed the importance of considering tensor products of pointed torsion free modules with finite dimensional modules

In an effort to gain some insight into such tensor product modules for C_n, we have constructed and classified all completely pointed C_n-modules and obtained some information regarding the decomposition of tensor products of such modules with finite dimensional modules. In particular, we have classified all of the possible simple highest weight submodules and all the possible central characters which can occur in such a tensor product. Hopefully, these results can provide some information regarding the tensor product of a pointed torsion free module with a finite dimensional module.

We are led to make several conjectures.

<u>Conjecture 1:</u> *Let M and N be the completely pointed C_n-modules having highest weights $-\frac{1}{2}\omega_2$ and $\omega_1 - \frac{3}{2}\omega_2$, respectively. Let $V(\lambda)$ be an irreducible finite dimensional module of highest weight $\lambda = \lambda_1\omega_1 + \cdots \lambda_n\omega_n$ $(\lambda_i \in \mathbb{Z}^+)$. Then the*

inequalities

$$0 \le a_1 \le 2\lambda_n + 1$$

$$0 \le a_i \le \lambda_{n-i+1} \qquad (2 \le i \le n),$$

along with the parity of $\sum_{i=1}^{n} a_i$, *give a complete listing of all maximal vectors and central characters which occur in* $M \otimes V$ *and* $N \otimes V$. ∎

<u>Conjecture 2:</u> *The tensor product of a completely pointed* C_n*-module with a finite dimensional module is completely reducible.* ∎

A C_n-module V is called **pointed** if V is irreducible and has a one dimensional weight space. V is called **torsion free** if every nonzero root vector of C_n acts injectively on V.

<u>Conjecture 3:</u> *The tensor product of a pointed, torsion free* C_n*-module with a finite dimensional module is completely reducible.* ∎

We wish to end with an example which illustrates the main results, and gives some credence to the first conjecture.

We restrict our attention to the Lie algebra C_2. We will make use of the symmetrizer contraction realization of simple finite dimensional representations as found in [BBL]. Consider the irreducible finite dimensional C_2-module $V(2\omega_1)$. This module can be realized as the submodule of $\mathbb{C}^4 \otimes \mathbb{C}^4$ generated by the vector $e_1 \otimes e_1$. Recall that in table 3.1 we explicitly listed the actions of C_2. In particular, $H_\alpha(e_1 \otimes e_1) = 2(e_1 \otimes e_1)$, $H_\beta(e_1 \otimes e_1) = 0$, and $X_\alpha(e_1 \otimes e_1) = X_\beta(e_1 \otimes e_1) = 0$. Hence $e_1 \otimes e_1$ is a maximal vector of weight $2\omega_1$.

4. Conclusion

By generating on $e_1 \otimes e_1$ we find a basis of $V(2\omega_1)$ comprised of weight vectors:

$$e_1 \otimes e_1$$

$$e_1 \otimes e_2 + e_2 \otimes e_1$$

$$e_2 \otimes e_2$$

$$e_1 \otimes e_4 + e_4 \otimes e_1$$

$$e_2 \otimes e_4 + e_4 \otimes e_2$$

$$e_2 \otimes e_4 + e_4 \otimes e_2 - e_1 \otimes e_3 - e_3 \otimes e_1$$

$$e_2 \otimes e_3 + e_3 \otimes e_2$$

$$e_4 \otimes e_4$$

$$e_3 \otimes e_4 + e_4 \otimes e_3$$

$$e_3 \otimes e_3$$

In particular, $V(2\omega_1)$ is 10-dimensional. In the following, we let V denote $V(2\omega_1)$.

Let us now consider $M \otimes V$. Using the conditions in equations (3.38), we see that if w^+ is a nonzero maximal vector in $M \otimes V$, with

$$w^+ = a x_1^{a_1} x_2^{a_2} \otimes v^+ + \cdots$$

and $a \neq 0$, then the exponents a_1, a_2 must satisfy

$$0 \leq a_1 \leq 1$$

$$0 \leq a_2 \leq 2$$

$$a_1 + a_2 \text{ even}$$

These conditions imply that (a_1, a_2) must equal $(0,0)$, $(1,1)$, or $(0,2)$. We wish to first show that Conjecture 1 holds for this module.

4. Conclusion

Case 1: $(a_1, a_2) = (0,0)$

In this case it is clear that $u = 1 \otimes (e_1 \otimes e_1)$ is a maximal vector, of highest weight $2\omega_1 - \frac{1}{2}\omega_2$.

Case 2: $(a_1, a_2) = (1,1)$

Consider the vector

$$v = 2x_1x_2 \otimes (e_1 \otimes e_1) - x_1^2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1) - 1 \otimes (e_1 \otimes e_4 + e_4 \otimes e_1).$$

An easy computation shows that $H_\alpha(v) = 2v$, $H_\beta(v) = -\frac{3}{2}v$, and $X_\alpha(v) = X_\beta(v) = 0$, so that v is a maximal vector of highest weight $2\omega_1 - \frac{3}{2}\omega_2$.

Case 3: $(a_1, a_2) = (0,2)$

Computations again immediately show that

$$\begin{aligned} w = {} & 2x_2^2 \otimes (e_1 \otimes e_1) - 2x_1x_2 \otimes (e_1 \otimes e_2 + e_2 \otimes e_1) + 2x_1^2 \otimes (e_2 \otimes e_2) \\ & + 2\,(1 \otimes (e_2 \otimes e_4 + e_4 \otimes e_2)) \\ & - (1 \otimes (e_2 \otimes e_4 + e_4 \otimes e_2 - e_1 \otimes e_3 - e_3 \otimes e_1)) \end{aligned}$$

is a maximal vector of weight $-\frac{1}{2}\omega_2$.

Hence the inequalities in (3.38), along with the parity of $\sum_{i=1}^{n} a_i$ give a complete listing of all maximal vectors which occur in $M \otimes V$. Using Theorem 3.46 we see also that this gives a complete listing of all irreducible standard cyclic submodules of $M \otimes V$ and all central characters occurring in $M \otimes V$, and so Conjecture 1 holds for $M \otimes V$.

Computer programs have shown that the second and third conjectures hold for finite dimensional modules of small dimension, in particular for the example $V(2\omega_1)$ treated above, but the calculations are too involved to include here.

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VITA AUCTORIS

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